On the ELzaki Transform and System of Partial Differential Equations

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Abstract

In this work a new integral transform, namely ELzaki transform was applied to solve linear system of partial differential equations with constant coefficients.

Keywords: Elzaki transform-system of partial differential equation.

Introduction

The system of differential equations have played a central role in every aspect of applied mathematics for every long time and with the advent of the computer, their importance has increased father.

Thus investigation and analysis of differential equations cruising in applications led to many deep mathematical problems; therefore, there are so many different techniques in order to solve differential equations.

In order to solve the system of differential equations, the integral transforms were extensively used and thus there are several words on the theory and applications of integral transforms such as the Laplace, Fourier, Mellin, Hankel and Sumudu, to name but a few. Recently, Tarig Elzaki introduced a new integral transform, named the ELzaki transform, and further applied it to the solution of ordinary, partial differential equations and system of partial differential equations.

In this paper we derive the formulate for ELzaki transform of partial derivatives and apply them to solving initial value problems. Our purpose here is to show the applicability of this interesting new transform and its effecting in solving such problems.

Definition and Derivations the ELzaki Transform of Derivatives

ELzaki transform of the function \( f(t) \) is defined as
\[
E\left[ f(t) \right] = T(\nu) = \nu \int_0^\infty f(t) e^{-\nu t} dt \quad , \quad t > 0 \quad , \quad \nu \in (-k_1, k_2)
\]

(1)

To obtain ELzaki transform of partial derivatives we use integration by parts as follows:

\[
E\left[ \frac{\partial f}{\partial t} (x,t) \right] = \int_0^\infty \nu e^{-\nu t} \frac{\partial f}{\partial t} dt = \lim_{p \to \infty} \int_0^p \nu e^{-\nu t} \frac{\partial f}{\partial t} dt = \lim_{p \to \infty} \left[ \nu e^{-\nu t} f(x,t) \right]_0^p - \left[ \frac{\partial}{\partial t} (\nu e^{-\nu t} f(x,t)) \right]_0^p = \frac{T(x,\nu)}{\nu} - v f(x,0)
\]

(2)

We assume that \( f \) is piecewise continuous and of exponential order.

Now

\[
E\left[ \frac{\partial f}{\partial x} \right] = \int_0^\infty \nu e^{-\nu t} \frac{\partial f}{\partial x} dt = \frac{\partial}{\partial x} \int_0^\infty \nu e^{-\nu t} f(x,t) dt
\]

Using the Leibnitz rule we find that:

\[
E\left[ \frac{\partial f}{\partial x} \right] = \frac{d}{dx} \left[ T(x,\nu) \right]
\]

(3)

Also we can find:

\[
E\left[ \frac{\partial^2 f}{\partial x^2} \right] = \frac{d^2}{dx^2} \left[ T(x,\nu) \right]
\]

(4)

To find \( E\left[ \frac{\partial^2 f}{\partial x^2} (x,t) \right] \) let \( \frac{\partial f}{\partial t} = g \) then:

By using equation (2) we have

\[
E\left[ \frac{\partial^2 f}{\partial t^2} (x,t) \right] = E\left[ \frac{\partial g(x,t)}{\partial t} \right] = E\left[ \frac{g(x,t)}{\nu} \right] - v g(x,0)
\]
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\[ E \left[ \frac{\partial^2 f(x,t)}{\partial t^2} \right] = \frac{1}{v^2} T(x,v) - f(x,0) - v \frac{\partial f(x,0)}{\partial t} \]  

(5)

We can easily extend this result to the nth partial derivative by using mathematical induction.

**Example I:**

Consider the general system of the first order partial differential equation.

\[
\begin{align*}
\alpha U(x,t) + \alpha V(x,t) &= a_i(x,t) \\
\beta U(x,t) + \beta V(x,t) &= a_2(x,t), \quad x, t > 0
\end{align*}
\]

(6)

With the initial conditions

\[
U(x,0) = b_i(x), \quad V(x,0) = b_2(x)
\]

(7)

Where that \( \alpha_i, a_i, \beta_i \) are constants. \( \forall i = 1,2 \).

Taking ELzaki transform of equation (6) we obtain,

\[
\begin{align*}
\alpha \overline{U} - \alpha V(x,0) + \alpha \overline{V}_x &= \overline{a}_i(x,v) \\
\beta \overline{U}_x + \beta \overline{V} - \beta \overline{V}(x,0) &= \overline{a}_2(x,v)
\end{align*}
\]

(8)

Substituting equation (7) into equation (8) yields,

\[
\begin{align*}
\alpha \overline{U} + \alpha \overline{V}_x &= \overline{a}_i + \alpha \overline{v}^2 b_i(x) \\
\beta \overline{U}_x + \beta \overline{V} &= \overline{a}_2 + \beta \overline{v}^2 b_2(x)
\end{align*}
\]

(9)

Or

\[
\begin{align*}
\alpha \beta \overline{U} + \alpha \beta \overline{V}_x &= \beta \overline{a} + \alpha \overline{v}^2 b_i(x) \\
\alpha \beta \overline{v}^2 \overline{U}_{xx} + \alpha \beta \overline{v} \overline{V}_x &= \alpha \overline{\overline{a}} \overline{v}^2 + \alpha \beta \overline{v}^3 b_2(x)
\end{align*}
\]

(10)

These equations are written as.
\[ \alpha \beta_1 v^2 \ddot{U} - \alpha \beta_1 \dot{U} = \alpha_2 \left( \ddot{a}_1 \right)_1 v^2 + \alpha \beta_2 v^3 b_2(x) \]
\[ - \beta_2 \ddot{a}_1 v - \alpha \beta_2 v^2 b_1(x) \]  
(11)

The solution of equation (11) is

\[ \bar{U} = \frac{\alpha \beta_1 v^2 \ddot{a}_1 + \alpha \beta_2 v^3 b_2(x) - \beta_2 \ddot{a}_1 v - \alpha \beta_2 v^2 b_1(x)}{\alpha \beta_1 v^2 D^2 - \alpha_2 \beta_2} = F(v)G(x) \]  
(12)

Apply inverse ELzaki transform to obtain \( U(x,t) \) in the form.

\[ U(x,t) = G(x) E^{-1}[F(v)] = G(x)f(t). \]

Assume that inverse ELzaki transform exists. In order to find \( V(x,t) \) substituting \( U(x,t) \) into equation (6).

**Example II:**

Consider the system given by the following first order initial value problem

\[ \begin{cases} U_x(x,t) + V_t(x,t) = 3x \\ 2U_x(x,t) - 3V_x(x,t) = t \end{cases} \]  
(13)

With the initial Conditions

\[ U(x,0) = x^2, V(x,0) = 0 \]  
(14)

**Solution**

By using ELzaki transform into equation (13) we have.

\[ \begin{cases} \ddot{U} + \frac{V}{v} - vV(x,0) = 3xv^2 \\ 2\ddot{V} - 2vU(x,0) - 3\dot{V}_x = v^3 \end{cases} \]  
(15)

Where that \( \bar{U} \) and \( \bar{V} \) are ELzaki transform of \( U \) and \( V \). We can write equations (15) in the form.
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\[
\begin{align*}
3v^2 \dddot{U} + 3v \dddot{V} &= 9v^4 \\
2\dddot{U} - 3v \dddot{V} &= v^4 + 2x^2v^2 
\end{align*}
\tag{16}
\]

Then

\[
3v^2 \dddot{U} + 2\dddot{U} = 10v^4 + 2x^2v^2 \quad \text{Or} \quad \dddot{U} = \frac{10v^4}{3v^2D^2 + 2} + \frac{2x^2v^2}{3v^2D^2 + 2}
\]

\[
\dddot{U} = 5v^4 + \left[1 + \frac{3}{2}v^2D^2\right]^{-1}x^2 \quad \text{And} \quad \dddot{U} = 5v^4 + v^2x^2 - 3v^4 = 2v^4 + v^3x^2
\]

By using inverse ELzaki transform with respect to \(v\) we obtain

\[
22
\]

Form the first equation of (13) we have

\[
V_t(x,t) = 3x - U_x = x \quad \text{or} \quad V(x,t) = xt + f(x)
\]

Substituting \(V(x,0) = 0\) we get \(f(x) = 0\) and thus \(V = xt\)

Examples III

Consider the following system

\[
\begin{align*}
z_t + w_x &= xe^t + e^{x+t} \\
z_x - w_t &= e^t - e^{x+t} 
\end{align*}
\tag{17}
\]

With the initial conditions

\[
z(x,0) = x, \quad w(x,0) = e^x
\tag{18}
\]

By using ELzaki transform into equation (17) and the initial Conditions (18) we have,

\[
\begin{align*}
\dddot{Z} + v \dddot{W} &= \frac{xv^2}{1-v} + \frac{e^xv^3}{1-v} \\
v^2 \dddot{Z} - v \dddot{W} &= \frac{v^3}{1-v} - \frac{v^2e^x}{1-v}
\end{align*}
\tag{19}
\]


\[
\begin{aligned}
\bar{Z} + v\bar{W} &= \frac{xy^2}{1-v} + e^t v^3 \\
\therefore v^2\frac{\partial^2 Z}{\partial x^2} - v \frac{\partial W}{\partial x} &= -\frac{v^3 e^t}{1-v}
\end{aligned}
\]

then we have:

\[
v\bar{Z}_{xx} + \bar{Z} = \frac{v^2}{1-v} x \quad \text{And} \quad \bar{Z} = \frac{v^2}{1-v} \frac{x}{vD^2 + 1} = \frac{xy^2}{1-v}
\]

In order to obtain the solution we apply the inverse ELzaki transform

\[z(x,t) = xe^t\]

From the first equation of (17) we get:

\[w(x,t) = e^{x-t}\] \hfill (20)

**Example (III):**

Consider the following system.

\[
\begin{cases}
\frac{\partial^2 z (x, y)}{\partial t^2} - \frac{\partial w (x, t)}{\partial x} = 2x^2 - e^t \\
\frac{\partial w (x, t)}{\partial t} + \frac{\partial^2 z (x, t)}{\partial x^2} = 2t^2 + xe^t
\end{cases}
\]

(21)

With the initial conditions

\[z(x, 0) = 0, \quad z_t (x, 0) = 0, \quad w(x, 0) = x\] \hfill (22)

By applying ELzaki transform to (21) and using the conditions (22) we get.

\[
\begin{cases}
\bar{Z} - v^2 \frac{\partial \bar{W}(x, y)}{\partial x} = 2x^2 v^4 - \frac{v^4}{1-v} \\
v^2 \frac{\partial \bar{W}(x, y)}{\partial x} + v^2 \frac{\partial^3 \bar{Z}(x, v)}{\partial x^3} = \frac{v^4}{1-v}
\end{cases}
\]

Then we have:
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\[ v^2 D^3 \tilde{Z} + \tilde{Z} = 2x^2v^4 \quad \text{And} \quad \tilde{Z} = \frac{2v^4x^2}{1 + v^2D^3} = 2v^4x^2 \]

Inverting to find that: \( z(x,t) = x^2t^2 \). To find \( w(x,t) \) we have:

\[ \frac{dw}{dx} = \frac{d^2z}{dt^2} + e^t - 2x^2 = e^t \Rightarrow w(x,t) = xe^t \]

**Examples V**

Consider the constant Coefficients system of partial differential equation in the form of:

\[
\begin{align*}
U_{xx} + V_y &= -2U \\
V_{xx} + U_y &= 0
\end{align*}
\]

(23)

With the initial conditions \( U(x,0) = \sin x, \ V(x,0) = \cos x \) (24)

**Solution:**

By using ELzaki transform and apply the initial condition (24) we have:

\[
\begin{align*}
\overline{U}_{xx} + \frac{\overline{V}}{v} - vV(x,0) &= 2\overline{U} \\
\frac{\overline{U}}{v} - vU(x,0) + \overline{V}_{xx} &= 0
\end{align*}
\]

(25)

Equations (25) can be written in the following form.

\[
\begin{align*}
v^2\overline{U}_{xxxx} + v\overline{V}_{xxx} + 2v^2\overline{U}_{xx} &= -v^3 \cos x \\
\overline{U} + v\overline{V}_{xx} &= v^2 \sin x
\end{align*}
\]

Then we have: \( v^2\overline{U}_{xxxx} + 2v^2\overline{U}_{xx} - \overline{U} = -v^2 \sin x - v^3 \cos x \) or

\[
\overline{U} = \frac{-v^2 \sin x - v^3 \cos x}{v^2D^3 + 2v^2D^2 - 1} = \frac{-v^2 \sin x - v^3 \cos x}{-v^2 - 1} = \frac{v^2 \sin x + v^3 \cos x}{1 + v^2} \]

By taking inverse ELzaki transform we have:

\[ U(x,y) = \sin x \cos t + \cos x \sin t = \sin(x + t) \]
From the first equation of (23) we have: \( V_y = -2U - U_{xx} = -\sin(x + y) \)

Then: \( V = \cos(x + y) + f(x) \)

Substituting \( V(x,0) = \cos x \) in the last equation we get \( f(x) = 0 \). Then:

\[ V(x,t) = \cos(x + y) \]

**Conclusions**

ELzaki transform provides powerful method for analyzing partial derivatives. It is heavily used in solving Partial differential equations and ordinary differential equations. The main purpose of this work is to solve the system of partial differential equations.

**References**

