On the Connections Between Laplace and ELzaki Transforms

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Abstract
This paper discusses some relationship between Laplace transform and the new transform Called ELzaki transform, we solve first and second order ordinary differential equations using both transforms, and showing The ELzaki transform is closely connected with The Laplace transform.

Keywords: ELzaki transform. Laplace transforms. Differential Equation.

Introduction
The definitions, Properties and applications of the new integral transform (ELzaki transform) to ordinary differential equations are described in this paper. The ELzaki transform can be used to solve ordinary differential equation and Control engineering Problems. The ELzaki transform easier than the Laplace transform for beginners to understand and use, also this transform can still serve as an auxiliary method to the Laplace transform.

A lot of work has been done on the theory and applications of transforms such as Laplace-Fourier-Melin- Hankel and Sumudu, to name a few, but very little on the power series transformation or ELzaki transform, probably because it is little known ,and not widely used, the ELzaki transform rivals the Laplace transform in problem Solving , its main advantage is the rivals that it many be used to solve problems without resorting to anew frequency domain because it preserve scales and units properties, the ELzaki transform may be used to solve intricate problems in engineering , mathematics and applied science without resorting to a new frequency domain.

The theory of ELzaki transform defined for functions of exponential order is adequate for many applications in mathematics, (ordinary and partial differential equations).

The ELzaki transform has deeper connections with the Laplace transform.
We recall that the Laplace is defined by

\[ L[f(t)] = \int_0^\infty e^{-st} f(t) \, dt, \quad \text{Re}(s) > 0 \]

While the ELzaki transform is defined by the following formulas.

\[ T(v) = \mathcal{E}[f(t), v] = v \int_0^\infty f(t) e^{-vt} \, dt, \quad v \in [-k_1, k_2] \quad (1) \]

or

\[ T(v) = \mathcal{E}[f(t)] = v^2 \int_0^\infty f(vt) e^{-vt} \, dt \quad k_1, k_2 > 0 \quad (2) \]

For any function, or by the conversion rule

\[ T(v) = \sum_{n=0}^{\infty} a_n v^{n+2} \quad (3) \]

For function \( f(t) \) which can be expressed as a polynomial or as a convergent in finite series for \( t > 0 \) applied it to the Solution of ordinary convergent equations and control engineering Problems.

The sufficient conditions for the existence of the ELzaki transform are that of exponential order. This means that the ELzaki transform may or may not exist.

\textbf{Theorem (1-1):}

Let \( T(v) \) is the ELzaki transform of \( f(t) \)

\[ \mathcal{E}(f(t)) = T(v) \quad \text{and} \quad g(t) = \begin{cases} f(t-\tau), & t \geq \tau \\ 0 & t < \tau \end{cases} \]

Then:

\[ \mathcal{E}[g(t)] = e^{-\tau v} T(v) \]

\textbf{Proof}

\[ \mathcal{E}[g(t)] = \int_0^\infty v e^{-\frac{vt}{\tau}} f(t-\tau) \, dt \]

Let \( t = \lambda + \tau \) we find that:

\[ \int_0^\tau v e^{-\frac{\lambda + \tau}{\tau}} f(\lambda) \, d\lambda = \]

\[ e^{-\frac{\tau}{\tau}} \int_0^\infty v e^{-\frac{\lambda}{\tau}} f(\lambda) \, d\lambda = e^{-\frac{\tau}{\tau}} T(v) \]

which is the desired result.
The ELzaki transform can certainly treat all problems that are usually treated by the well-known and extensively used Laplace transform.

Indeed as the next theorem shows the ELzaki transform is closely connected with the Laplace transform $F(s)$.

**Theorem (1-2):**

Let

$$f(t) \in A = \left\{ f(t) \right\}_{\exists M, k_1, k_2 > 0, \text{ such that}} \left\{ f(t) \right\}_{\text{if } t \in (-1)^j \times [0, \infty)}$$

With Laplace transform $F(s)$, then the ELzaki transform $T(v)$ of $f(t)$ is given by

$$T(v) = vF\left(\frac{1}{v}\right)$$ (4)

**Proof**

Let: $f(t) \in A$. Then for $-k_1 < v < k_2$, $T(v) = \int_0^\infty e^{-vt} f(vt) dt$

Let $w = vt$ then we have:

$$T(v) = v^2 \int_0^\infty e^{-w} f(w) \frac{dw}{v} = \int_0^\infty e^{-w} f(w) dw$$

$$= vF\left(\frac{1}{v}\right).$$

Also we have that $T(1) = F(1)$ so that both the ELzaki and Laplace transforms must coincide at $v = s = 1$.

In fact the connection of the ELzaki transform with the Laplace transform goes much deeper, therefore the rules of $F$ and $T$ in (4) can be interchanged by the following corollary.

**Corollary (1-3)**

Let $f(t)$ having $F$ and $T$ for Laplace and ELzaki transforms respectively, then:

$$F(s) = sT\left(\frac{1}{s}\right)$$ (5)

**Proof**

This relation can be obtained from (4) by taking $v = \frac{1}{s}$

The equations (4) and (5) form the duality relation governing these two transforms and may serve as a mean to get one from the other when needed.
ELzaki Transform of Derivatives and Integrals

Being restatement of the relation (4) will serve as our working definition, since the Laplace transform of \( \sin t \) is \( \frac{1}{1+s^2} \) then view of (4), its. ELzaki transform is \( \mathcal{E}[\sin t] = \frac{v^3}{1+v^2} \) this exemplifies the duality between these two transforms.

**Theorem (2-1)**

Let \( F'(s) \) and \( T'(v) \) be the Laplace and ELzaki transforms of the derivative of \( f(t) \).

Then:

(i) \( T'(v) = \frac{T(v)}{v} - v f(0) \) \hspace{1cm} (6)

(ii) \( T^{(n)}(v) = \frac{T(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n-k} f^{(k)}(0) \) \hspace{1cm} (7)

Where \( T^n(v) \) and \( F^n(s) \) are the ELzaki and Laplace transforms of the nth derivative \( f^{(n)}(t) \) of the function \( f(t) \).

**Proof:**

(i) Since the Laplace transform of the derivatives of \( f(t) \) is \( F'(s) = sF(s) - f(0) \)

Then

\[
T'(v) = v F'(\frac{1}{v})
\]

\[
= v \left[ \frac{1}{v} F\left(\frac{1}{v}\right) - f(0) \right] = F\left(\frac{1}{v}\right) - v f(0)
\]

\[
= \frac{T(v)}{v} - v f(0)
\]

(ii) By definition, the Laplace transform for \( f^{(n)}(t) \) is given by

\[
F^{(n)}(s) = S^n F(s) - \sum_{k=0}^{n-1} S^{n-(k+1)} f^{(k)}(0)
\]

Therefore

\[
F\left(\frac{1}{v}\right) = \frac{F\left(\frac{1}{v}\right)}{v^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{n-(k+1)}}
\]
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Now, since
\[ T^{(k)}(v) = v F^{(k)} \left( \frac{1}{v} \right) \text{ for } 0 \leq k \leq m, \] we have
\[ T^{(n)}(v) = \frac{T(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n-k} F^{(k)}(0) \]

Theorem (2-2)
Let \( T'(v) \) and \( F'(s) \) denote the ELzaki and the Laplace transforms of the definite integral of \( f(t) \).
\[
h(t) = \int_0^t f(\tau) d\tau. \quad \text{Then} \]
\[ T'(v) = E \left[ h(t) \right] = vT(v) \]

Proof
By the definition of Laplace transform \( F'(s) = L(h(t)) = \frac{F(s)}{s} \)
Hence
\[ T'(v) = v F' \left( \frac{1}{v} \right) = F \left( \frac{1}{v} \right) = vT(v) \]

Theorem (2-3)
Let \( T(v) \) is the ELzaki transform of the function \( f(t) \) then:
(i) \[ E \left[ tf(t) \right] = v^2 \frac{dT(v)}{dv} - vT(v) \]

(ii) \[ E \left[ t^2 f(t) \right] = v^4 \frac{d^2}{dv^2}T(v) \]

Proof
From the definition of ELzaki transform we have.
\[ \frac{dT}{dv} = T'(v) = \frac{d}{dv} \int_{0}^{\infty} \nu e^{-\nu t} f(t) \, dt = \]

\[ \int_{0}^{\infty} \frac{d}{dv} \nu e^{-\nu t} f(t) \, dt = \int_{0}^{\infty} \frac{1}{\nu} e^{-\nu t} (t f(t)) \, dt + \int_{0}^{\infty} e^{-\nu t} f(t) \, dt \]

Then

\[ = \frac{1}{\nu^2} E(t f(t)) + \frac{1}{v} E(f(t)) \]

\[ E(tf(t)) = \nu^2 \frac{dT(v)}{dv} - vT(v) \]

Where that:

\[ E(f(t)) = T(v) \]

The proof of (ii) is similar

**Theorem (2-4)**

If \( E[f(t)] = T(v) \) then:

\[ (i) \, E[tf'(t)] = \nu^2 \frac{d}{dv} \left[ \frac{T(v)}{\nu} - \nu f(0) \right] - \nu \left[ \frac{T(v)}{\nu} - \nu f'(0) \right] \]

\[ (ii) \, E[tf''(t)] = \nu^2 \frac{d}{dv} \left[ \frac{T(v)}{\nu^2} - f'(0) - \nu f''(0) \right] - \nu \left[ \frac{T(v)}{\nu^2} - f(0) - \nu f'(0) \right] \]

\[ (iii) \, E[t^2f''(t)] = \nu^4 \frac{d^2}{dv^2} \left[ \frac{T(v)}{\nu^2} - f(0) - \nu f'(0) \right] \]

**Proof**

(i) From theorem (2-3) we have

\[ E[tf'(t)] = \nu^2 \frac{d}{dv} \left[ E(f'(t)) \right] - \nu E[f'(t)] = \]

\[ \nu^2 \frac{d}{dv} \left[ \frac{T(v)}{\nu} - \nu f(0) \right] - \nu \left[ \frac{T(v)}{\nu} - \nu f(0) \right] \]

The proof of (ii) and (iii) are similar to that of (i)

**Theorem (2-5) (shift)**

Let \( f(t) \in A \) with Elzaki transform \( T(v) \). Then
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\[ E\left[ e^{at} f(t) \right] = \frac{1}{1-av} \quad T\left[ \frac{v}{1-av} \right] \]

**Proof**

From definition of ELzaki transform we have:

\[ E\left[ e^{at} f(t) \right] = v^2 \int_0^\infty f(vt) e^{-\left(1-av\right)t} dt \]

Let \( w = (1-av)t \Rightarrow dw = (1-av)dt \)

Then

\[
\frac{v^2}{1-av} \int_0^\infty f\left(\frac{wv}{1-av}\right) e^{-w} \, dw = \frac{1}{1-av} \quad T\left[ \frac{v}{1-av} \right]
\]

**Theorem (2-6) (convolution)**

Let \( f(t) \) and \( g(t) \) be defined in \( A \) having Laplace transforms \( F(s) \) and \( G(s) \) and ELzaki transforms \( M(v) \) and \( N(v) \). Then the ELzaki transform of the convolution of \( f \) and \( g \)

\[(f * g)(t) = \int_0^\infty f(t)g(t-\tau)\,d\tau \text{ is given by:}\]

\[ E\left[ (f * g)(t) \right] = \frac{1}{v} M(v) N(v) \]

**Proof**

The Laplace transform of \((f * g)\) is given by:

\[ L\left[ (f * g) \right] = F(s) G(s) \]

By the duality relation (4) we have:

\[ E\left[ (f * g)(t) \right] = v L\left[ (f * g)(t) \right] , \]

and since

\[ M(v) = v F\left( \frac{1}{v} \right) , \quad N(v) = v G\left( \frac{1}{v} \right) \]

Then

\[ E\left[ (f * g)(t) \right] = v \left[ F\left( \frac{1}{v} \right) G\left( \frac{1}{v} \right) \right] \]

\[
\frac{v}{v} M(v) \cdot \frac{N(v)}{v} = \frac{1}{v} M(v) N(v)
\]
Solution of Ordinary Differential Equations.

Example (3-2)
Consider the first order Differential equation:
\[
\frac{dx}{dt} + px = f(t), \quad t > 0
\]
(8)

With the initial Condition \( x(0) = a \)

Where \( p \) and \( a \) are Constants and \( f(t) \) is an external input function so that its Laplace transform and ELzaki transform exist

First Solution by Laplace transforms
\[
s \bar{x}(s) - x(0) + p \bar{x}(s) = \bar{f}(s)
\]

\[
\bar{x}(s) = \frac{a}{s + p} = \frac{\bar{f}(s)}{s + p}
\]

Where that \( \bar{x}(s) \) and \( \bar{f}(s) \) are Laplace transform of \( x(t) \) and \( f(t) \). Then

\[
x(t) = ae^{-pt} + \int_0^t f(t - \tau) e^{-p\tau} d\tau
\]

Thus the solution naturally splits into two terms the first one Corresponds to the response of the initial condition and the second one is entirely due to the external input function \( f(t) \). In particular if \( f(t) = c \) \( c \) is the constant then the Solution of (8) becomes

\[
x(t) = c + \left( a - \frac{c}{p} \right) e^{-pt}
\]
(9)

Second Solution by ELzaki Transfor
Using ELzaki transform of equation (8) we have:
\[
\frac{X(v)}{v} - v x(0) + PX(v) = F(v)
\]

By applying the initial condition we have
\[
X(v) = \frac{vF(v)}{1 + vP} + \frac{av^2}{1 + vP}
\]
(10)

Where that \( X(v) \) and \( F(v) \) are ELzaki transform of \( x(t) \) and \( f(t) \).

By the inverse of ELzaki transform and Convolution theorem (2 – 6) we find that:
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\[ x(t) = a e^{-pt} + \int_{0}^{t} f(t-\tau) e^{-p\tau} d\tau \] (11)

Which is the same solution

The first term of this solution in (9) is independent of time \( t \) and is usually called the steady – state solution, and the second term depends on time \( t \) and is called the transient solution. In the limit as \( t \to \infty \) the transient solution decays to zero if \( p > 0 \) the steady – state solution is attained, on the other hand, when \( p < 0 \), the transient solution grows exponentially as \( t \to \infty \) and the solution becomes unstable.

Equation (8) describes the Law of natural growth or decay Process with an external forcing function \( f(t) \) according as \( p > 0 \) or \( P < 0 \).

In particular, if \( f(t) = 0 \) and \( P > 0 \)

The resulting equation (8) occurs very frequently in chemical Kinetics. Such an equation describes the rate of chemical reactions.

**Example (3-2)**
Consider the Ordinary differential equation with Variable Coefficients:

\[ ty'' - ty' + y = 2 \quad , \quad y(0) = 2 \quad , \quad y'(0) = -1 \] (12)

**Solution by Laplace Transform**
Using Laplace transform and apply the Conditions, we have

\[
\frac{d}{ds} Y(s) + \left( \frac{2}{s} \right) Y(s) = \frac{2}{s^2} \\
\left[ L(y(t)) = Y(s) \right]
\]

This is a linear differential equation and the solution is

\[ Y(s) = \frac{2}{s} + \frac{c}{s^2} \quad [c \equiv constant] \]

Take the inverse Laplace transform we find that:

\[ y(t) = 2 + ct \]

**Solution by ELzaki Transform**
Take the ELzaki transform of equation (12) we have

\[
v^2 \frac{d}{dv} \left[ T(v) \right] - y(0) - vy'(0) - v \left[ \frac{T(v)}{v^2} - y(0) - vy'(0) \right] - \\
v^2 \frac{d}{dv} \left[ \frac{T(v)}{v} \right] - vy(0) + v \left[ \frac{T(v)}{v} - vy(0) \right] + T(v) = 2v^2
\]

\[ \left[ E(y(t)) = T(v) \right] \]

Now Applying the initial condition to obtain:
\[(1-v)T'(v) + 3\left(1 - \frac{1}{v}\right)T(v) = -v^2 - 2v\]

Or
\[T'(v) - \frac{3}{v}T(v) = -\frac{v^2 - 2v}{1-v}\]

Equation (13) is a linear differential equation, which has solution in the form
\[T(v) = cv^3 + 2v^2\quad [c \equiv \text{cons tan } t]\]

Using inverse ELzaki transform to find
\[y(t) = ct + 2,\]
Which is the same Solution

**Conclusions**
The origin of ELzaki transform is traced back to the classical Fourier integral. ELzaki transform is a convenient tool for solving differential equations in the time domain without the need for performing an inverse ELzaki transform.

The connection of the ELzaki transform with the Laplace transform goes much deeper.

**Reference**