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Estimation of Aquifer Parameters Using the Numerical Inversion of Laplace Transform

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Abstract

A new method is presented for the estimation of parameters for a circular aquifer by nonlinear regression analysis using numerical inversion of Laplace transform. The parameters estimated are the relative aquifer size R_{eD}, the storativity $h\phi C_t$ and the transmissibility kh/μ . These parameters are necessary to calculate water influx needed in performance prediction of oil and/or gas reservoirs by material balance based methods.

Water influx data are fitted to the van-Everdingen and Hurst (VEH) unsteady state solution to obtain the required aquifer parameters by nonlinear regression analysis using the method of least squares. Because the solution in Laplace space is simpler than the solution in the real time domain, numerical inversion of Laplace transform was used to obtain the partial derivatives of the VEH solution with respect to aquifer parameters needed for least squares method. The Levenberg method was used for parameter estimation to guarantee convergence. This procedure proved to be efficient and free of computational and convergence problems encountered when using real time solution.

Two approaches are used to represent the variable pressure history: the step pressure SP and the linear pressure LP methods.The two approaches are used to generated water influx data and the values obtained by both methods are compared with the actual (assumed) data. The LP method is found to yield more accurate results and is used in the parameter estimation algorithm.

The developed algorithm can be applied for performance prediction of oil and gas reservoirs under water drive and for the simultaneous estimation of original hydrocarbons in place (OHIP) and aquifer parameters based on material balance equations.

Introduction

The estimation of initial hydrocarbons (oil and/or gas) in place OHIP is of great importance for future development of these reservoirs. Volumetric methods based on geological and seismic .data can be used at early stages of development. With a reasonable estimates of OHIP, the material balance equation MBE can be used to predict future reservoir performance. If enough production data are available for a given reservoir, the MBE can be used to estimate the OHIP. For volumetric reservoirs (no water influx), the MBE is linear the parameters N and G. In this case, the MBE represent an equation of a plane. Havlena and Odeh[1] showed how it can be arranged as an equation of a straight line by grouping production and pressure dependent terms. Tehrani[2], however, indicated that regression should be performed on the original (non-grouped) MBE to preserve the physical meaning of regression variables. In these cases, linear or multiple regression analysis by the method of least squares is used to estimate the original oil in place N and the gas cap ratio m for oil reservoirs or the original gas in place G_i for gas reservoirs.

For non-volumetric (water drive) reservoirs the material balance equation can be used to estimate both OHIP and aquifer parameters. An aquifer model describing water influx from the aquifer into the reservoir is needed. In most field cases such model is nonlinear. The Van-Evedingen and Hurst's (VEH) unsteady state model[3] is an exact analytical solution for circular aquifers with homogeneous properties. The model is linear with respect to the water influx constant B, but nonlinear with respect the dimensionless aquifer size R_{eD} and time adjustment factor c which transforms real time t into dimensionless time t_D. It is therefore obvious that linear regression can not be used directly to estimate both OHIP and aquifer parameters B, c, and R_{eD}.

To overcome the nonlinearity problem, most investigators used some kind of a trial and error approach. In such cases values for aquifer parameters are assumed and linear regression is performed to estimate N and m for oil or G_i for gas reservoirs. The standard deviation or the sum of squares of errors is calculated. The values of aquifer parameters are changed and the process is repeated and the values of parameters that yield the smallest sum of squares of errors are selected. Most investigators also used the tabulated values of $Q(t_D)$ given by Van-Everdingen and Hurst to calculate the water influx.

Dougherty *et al.*[4] used tabulated values of the solution obtained by Wattenbarger and Ramey[5] to get the error terms. By performing different runs within the limits of parameters, they expressed the sum of squares of errors as a second order polynomial in the parameters and performed the minimization to obtain the optimum values of the parameters.

Chen, Chen and lin[6] used polynomial approximations of $Q(t_D)$ by Klins *et al.*[7] to evaluate the water influx and used the simplex search method for parameter estimation.

A different approach used by other investigators[8-9] is the aquifer influence function AIF which is the reservoir pressure response to a unit rate of water influx. It is directly proportional to the dimensionless pressure solution for constant rate. It is however treated as a general function regardless of aquifer geometry and homogeneity. It can thus provide a function for estimating water influx from pressure history. However, the extrapolation outside the production data range is needed for prediction of future performance. This extrapolation is questionable unless the AIF develops a clear trend within the available time which should be large enough to reach the trend. Different types of aquifers develop different trends and so knowledge of the shape of the aquifer is needed which limits the general applicability of this approach since it is meant to handle cases of unknown or irregular shapes.

Chatas and Malekfan[10] proposed the application of nonlinear regression to the Van-Everdingen and Hurst solution in real space to estimate aquifer parameters. They outlined the procedure and evaluated the partial derivatives but did not present actual solutions. The VEH solution in real time space is very complex and computational and convergence problems may be encountered. The solution in Laplace space is simpler and the use of the numerical inversion of the Laplace transform makes it easier to evaluate the first and second derivatives with respect to the aquifer parameters. This approach will be applied in this paper.

Theoritical Considerations

The conventional form of the MBE can be following in the following form

$$N_p(\beta_{oc} - R_s\beta_g) + G_p\beta_g + W_p\beta_w = N\left[\beta_o - \beta_{oi} + (R_{si} - R_s)\beta_g\right] + G_i(\beta_g - \beta_{gi}) + We \quad \text{……...………………… (1)}$$

Equation (1) can be used for undersaturated oil reservoirs with $G_i = 0$ or for gas reservoirs with N=0.

In simple models, the water influx from the aquifer into the reservoir is treated using Schilthuis steady state model.

$$We = K\int_0^t (p_i - p)dt \quad \text{………………………………… (2)}$$

This makes the material balance equation linear in the parameters N, G_i and K taking the following form.

$$y = N\,X_1 + G_i\,X_2 + K\,X_3 \quad \text{………………………….. (3)}$$

Equation (3) represents a hyper plane and multiple regression analysis can be used to estimate the three parameters N , G_i and K using production and PVT data.

The steady state water influx model however rarely describes the actual behavior of aquifers. An exact mathematical solution of the diffusivity equation is given by van Everdingen and Hurst[3] for a radial flow system at constant terminal pressure. The solution is given by

$$We = B\Delta pQ(t_D) \quad \text{…………… ………………… (4)}$$

where for We in Bbl, h and r_w in ft , K in md and t in days

$$B = 1.119h\varphi\, C_t r_w^2 \quad \text{…………………..………… (5)}$$

$$t_D = \frac{.00634Kt}{\mu\, C_t\varphi\, r_w^2} = ct \quad \text{………..……..……………. (6)}$$

The dimensionless function $Q(t_D)$ is given by

$$Q(t_D) = \frac{R_{eD}^2 - 1}{2} - 2\sum_{n=1}^{\infty}\frac{J_1^2(\alpha_n R_{eD})e^{-\alpha_n^2 t_D}}{\alpha_n^2\left[J_0^2(\alpha_n) - J_1^2(\alpha_n R_{eD})\right]} \quad \text{.... (7)}$$

where α_n are roots of the equation

$$J_1(\alpha_n R_{eD})Y_0(\alpha_n) - Y_1(\alpha_n R_{eD})J_0(\alpha_n) = 0 \quad \text{……..…………… (8)}$$

For variable pressure case, the principle of superposition (convolution) is applied to estimate the cumulative water influx into the reservoir. The solution is given by

$$We(t_D) = B\int_0^{t_D} Q(t_D - \tau)\Delta P'(\tau)d\tau \quad \text{……………….. (9)}$$

Aquifer Parameters

From Eq. (4)-(7) , the parameters needed to evaluate the water influx We at a given time t are the aquifer constant B, the relative aquifer radius $R_{eD}=r_e/r_w$ and the constant c in Eq. (6) which transforms real time t into dimensionless time t_D.

The parameters B and c can be expressed in terms of the commonly used parameters of transmissibility T (Kh/μ) and storativity S ($h\phi c_t$) as follows

$$B = 1.119h\phi\, C_t r_w^2 = 1.119Sr_w^2 \quad \text{……..…..…. (10)}$$

$$c = .00634\frac{k}{\mu\phi\, C_t r_w^2} = .000264\frac{T}{Sr_w^2} \quad \text{…… (11)}$$

So the transmissibility T and storativity S can be obtained from parameters B and C as follow

$$T = 141.03Bc \quad \text{………..……….……………. (12)}$$

$$S = .8936\frac{B}{r_w^2} \quad \text{…………………..………………. (13)}$$

Moreover, the parameter B is related to the reservoir pore volume V_p as follow

$$B = 2C_t V_p \quad \text{…………………..………………………. (14)}$$

Sills[11] related the parameter B to the original oil in place N

$$B = 2C_t V_p = 2C_t\frac{N\beta_{oi}}{(1 - S_{wi})} \quad \text{…..……………….. (15)}$$

Similar relations can also be written for the cases of gas reservoirs or oil reservoirs with a gas cap.This relation reduces the number of parameters to be estimated by one parameter by making B and N , G, or $N(1+m)$ dependent. Although this may sound attractive mathematically, its physical validity is questionable. The term V_p in Eq. (14), (15) represent all the pore volume present in the reservoir within the radiud r_w and thickness h including any shales and nonconnected pore volume. The term N in the material balance equation represents the oil subjected to expansion due to pressure change which includes only oil present in interconnected pore space. It is therefore important to have B and N or G_i as two independent variables when applying regression analysis for parameter estimation to the material balance equation.

Numerical Inversion of Laplace Transform

The complexity of using equation (7) in computing the water influx We is apparent. First Eq. (8) must be solved iteratively for enough numbers of successive roots α_n. The summation in Eq.(7) is to be continued until convergence of the infinite series is achieved. The problem is further complicated in the parameter estimation problem where in addition to evaluating We, the derivatives of $Q(t_D)$ must also be evaluated. These difficulties promp ted the investigation of the possibility of performing evaluation and optimization in Laplace spase using the Stehfest algorithm[12] for the numerical inversion of Laplace transform.

In Laplase space, the expression for the transform of the dimensionless water influx $\overline{Q}(s)$ is

$$\overline{Q}(s) = \frac{I_1(\sqrt{s}R_{eD})K_1(\sqrt{s}) - K_1(\sqrt{s}R_{eD})I_1(\sqrt{s})}{s^{3/2}\left[K_1(\sqrt{s}R_{eD})I_0(\sqrt{s}) + I_1(\sqrt{s}R_{eD})K_0(\sqrt{s})\right]} \quad \text{........................ (16)}$$

The inverse of $\overline{Q}(s)$ by the stehfest algorithm is

$$Q(t_D) = l^{-1}[\overline{Q}(s)] = \frac{\ln 2}{t_D}\sum_{i=1}^{N} V_i \overline{Q}(s = \frac{i\ln 2}{t_D}) \quad \text{... (17)}$$

To test the validity of Eq.(17), values of $Q(t_D)$ were calculated for values of R_{eD} of 2, 3, 4, 5 and 10 at different values of t_D. The results are shown in Table 1 and plotted in Fig.1.Comparison with tables given by van-Everdingen and Hurst shows very good agreement (to within 1%). It is known that the maximum (ultimate) value of Q (t_D) is $0.5(R_{eD} - 1)$. These values are 1.5, 4, 7.5, 12, 49.5for the reported R_{eD} values respectively. Values in table 1 show an oscillation about these values of $\pm$ 0.005 at large values of t_D.

Calculations of *We* Using SP and LP methods

For variable reservoir pressure,the pressure history is approximated into a number of constant pressure steps with discontinuous jumps at the data points as shown in fig. 2. The integration in Eq. (9) is then approximated by a summation as follows

$$We(k) = B\sum_{j=1}^{k}\Delta P_j\, Q[t_D(k) - t_D(j-1)] \quad \text{............... (18)}$$

This method is called the step pressure (SP) method.

Table 1: Generated Dim. Water Influx Q(t_D)

Dim. time t_D	Dim. Water Influx Q (t_D)				
	R_{eD} = 2	R_{eD} = 3	R_{eD} = 4	R_{eD} = 5	R_{eD} = 10
0.01	0.11778	0.11778	0.11778	0.11778	0.11778
0.05	0.27638	0.27637	0.27637	0.27637	0.27637
0.1	0.40421	0.40436	0.40436	0.40436	0.40436
0.2	0.59737	0.59799	0.59798	0.59797	0.59797
0.5	0.98396	1.02406	1.02453	1.02454	1.02452
1	1.29377	1.56361	1.56741	1.56799	1.56802
2	1.46592	2.35593	2.44180	2.44467	2.44576
5	1.50151	3.48798	4.32781	4.50414	4.53464
10	1.50026	3.92421	6.03503	6.97577	7.40021
20	1.50029	4.00395	7.17585	9.73213	12.29633
30	1.49976	4.00253	7.42979	10.95770	16.56406
40	1.50028	4.00178	7.49304	11.51922	20.34664
50	1.49947	3.99817	7.50276	11.77618	23.68692
60	1.50039	4.00068	7.50826	11.90499	26.66133
70	1.50006	4.00045	7.50532	11.96341	29.28194
80	1.50041	4.00091	7.50522	11.99443	31.60595
90	1.50015	3.99981	7.50261	12.00521	33.65370
100	1.49966	3.99785	7.49927	12.00849	35.45978

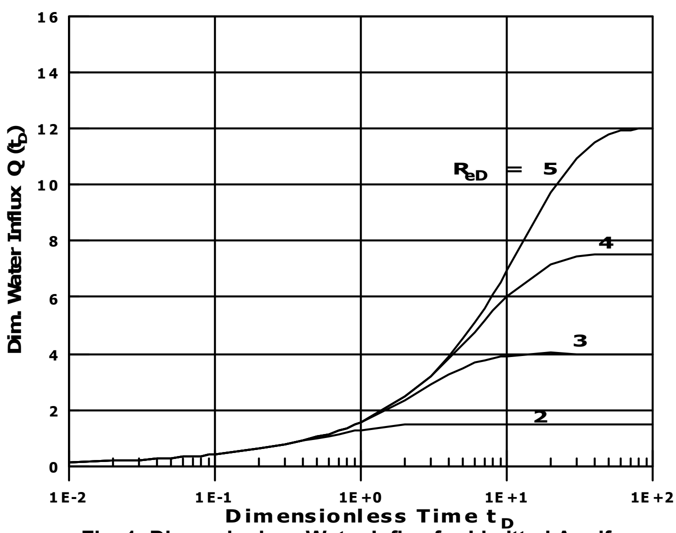

Fig. 1: Dimensionless Water Influx for Limitted Aquifers

Vogt and Wang[13] suggested approximating the pressure behavior by a series of linear segments connecting successive data points. This is expected to give a more accurate representation of the pressure history. The basis for this method is to replace $\Delta P'$ in Eq. (9) by the slope m and integrating by parts to obtain

$$We(t_D) = B\int_0^{t_D}\widetilde{Q}(t_D - \tau)\frac{dm}{d\tau}(\tau)d\tau \quad \text{................. (19)}$$

where

$$\widetilde{Q}(t_D) = \int_0^{t_D} Q(\tau)d\tau \quad \text{.................................. (20)}$$

Equation (19) is then approximated by a summation to

$$We(k) = B\sum_{j=1}^{k}\Delta m_j\, \widetilde{Q}[t_D(k) - t_D(j-1)] \quad \text{......... (21)}$$

This method is called thr linear pressure (LP) method. It is to be noticed that the slope m is calculated w.r.t the dimensionless time t_D

In Laplace space, using rules for transform of the integrals,

$$\overline{\widetilde{Q}}(s) = l[\widetilde{Q}(t_D)] = \frac{\vec{Q}(s)}{s} \quad \text{.......................... (22)}$$

where $\overline{Q}(s)$ is given by Eq. (16)

To test and compare the SP and LP methods, the pressure drop was generated for the case of a constant water influx rate i.e. $We = q\ t$ with q constant.

The following data were used to generate the pressure behavior:

Permeability K = 100 md. Thickness h = 20 ft.
Porosity f = 20% Viscosity m = 1 cp.
Reservoir radius r_w =1000 ft Aqifer rad. ratio R_{eD} = 5.
Compressibility C_t = 5 E -6 psi^{-1}
Water influx rate = 1000 res. Bbl/day

From this data, the following values are obtained
Transmissibility $T = Kh/m$= 2000 md. ft./cp.
Storativity $S = h\ f c_t$ = 2 E -5 ft. / psi
B = 1.119 s r_w^2 = 22.38 Bbl/psi
c = 0.0264 hr^{-1} = 0.6336 day^{-1}

The dimensionless pressure is obtained from van Everdingen and Hurst solution for the case of a closed finite reservoir with constant terminal rate

$$\overline{P}_{wD}(s) = \frac{K_1(\sqrt{s}R_{eD})I_0(\sqrt{s}) + I_1(\sqrt{s}R_{eD})K_0(\sqrt{s})}{s^{3/2}\left[I_1(\sqrt{s}R_{eD})K_1(\sqrt{s}) - K_1(\sqrt{s}R_{eD})I_1(\sqrt{s})\right]} \quad \text{... (23)}$$

$$P_{wD} = l^{-1}[\overline{P}_{wD}(s)] \quad \text{...................................... (24)}$$

.Results for the given reservoir and fluid data are evaluated using the numerical inversion algorithm of Laplace transform and compared with those given in table 3 of reference3. The actual pressure drop is obtained from the dimensionless pressure using the relation

$$P_{wD} = \frac{T\Delta p}{141.3q} \quad \text{.. (25)}$$

The obtained pressure drop vs. time is shown in fig. 2. The pressure steps for the SP method and the linear segments for the LP method are shown in Fig, 3. Water influx was calculated by the two methods using Eq. (18) and (21) and evaluating Q and $\widetilde{Q}$ by the stehfest algorithm applying Eq. (16), (17) and (22). The results are given in Table 2 and shown in Fig. 4 together with the actual (assumed) linear data. The absolute error in We for both methods are shown in Fig. 5. The results indicate that the LP method is more accurate than the SP method and will be used exlusively for parameter estimation.

Table 2: Generated Data for Constant Rate

Time (days)	DP (psi)	We (res. Bbl)		
		LP	SP	True
1	47.7807	794.72	631.04	1000
2	61.7195	1851.40	1786.99	2000
3	71.0257	2872.07	2824.04	3000
4	78.0979	3882.19	3843.58	4000
5	83.8708	4882.26	4854.78	5000
10	105.8377	9844.55	9699.63	10000
20	143.5380	19877.88	19454.55	20000
30	180.8897	29960.03	29396.11	30000
40	218.2472	40029.70	39340.54	40000
50	255.5763	50022.52	49293.98	50000
60	292.9779	60082.29	59270.38	60000
70	330.1909	69917.18	69248.75	70000
80	367.4729	79745.91	79225.88	80000
90	404.9226	90169.29	89225.96	90000
100	442.2198	100118.20	99232.91	100000

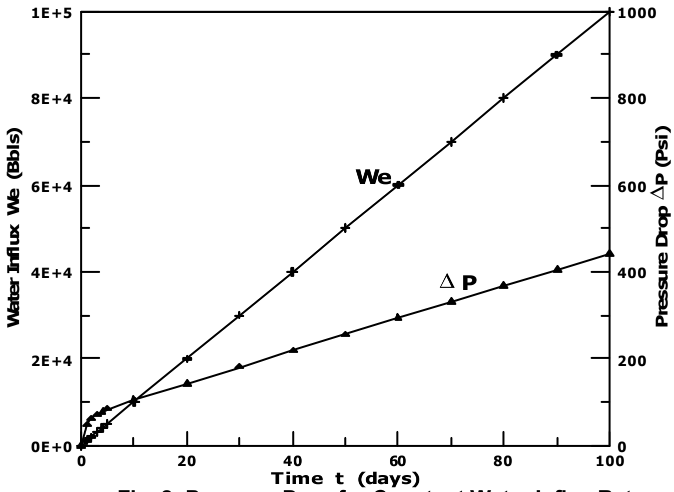

Fig. 2: Pressure Drop for Constant Water Influx Rate

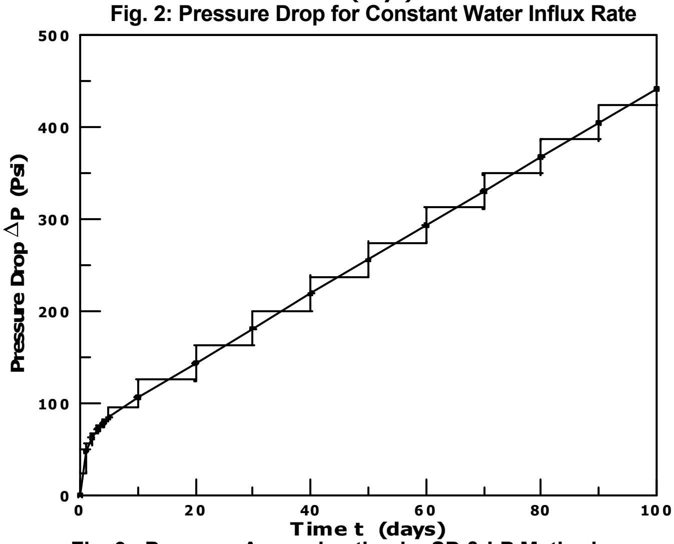

Fig. 3 : Pressure Approximation by SP & LP Methods

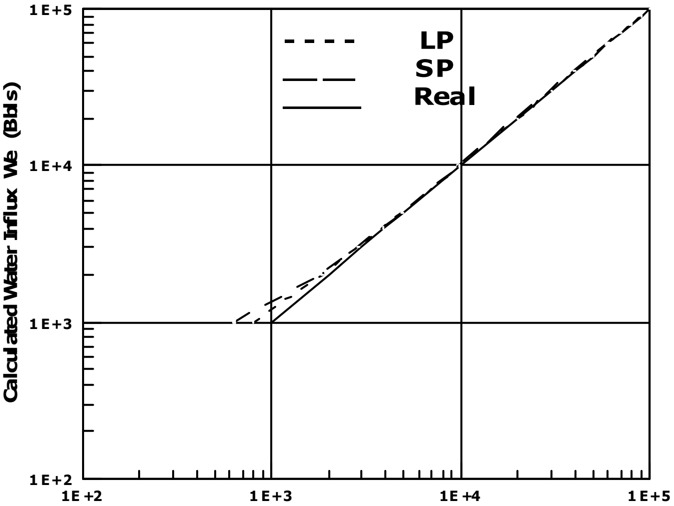

Fig. 4: Generated Water Influx By SP & LP Methods

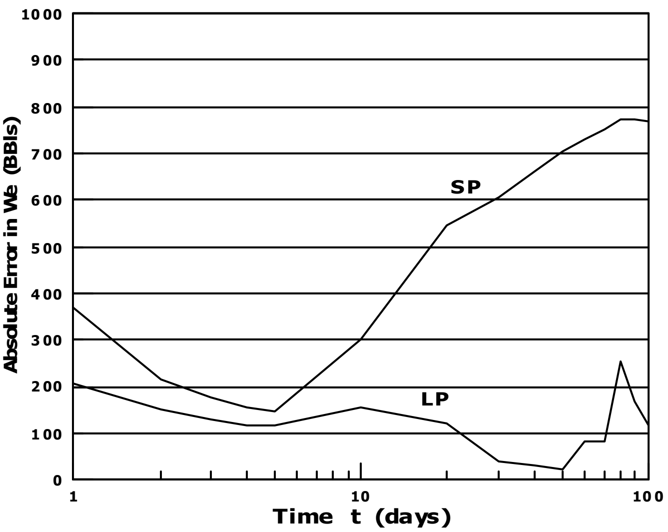

Fig. 5 : Absolute Errors in We by SP and LP Methods

Aquifer Parameter Estimation

The material balance equation can be written in the following form

$$N_p(\beta_{oc} - R_s\beta_g) + G_p\beta_g + W_p\beta_w = N[\beta_o - \beta_{oi} + (R_{si} - R_s)\beta_g] + G_i(\beta_g - \beta_{gi}) + B\sum_{j=1}^{k}\Delta P_j \frac{\ln 2}{c[t_k - t_{j-1}]}\sum_{i=1}^{N} V_i\overline{Q}(s = \frac{i\ln 2}{c[t_k - t_{j-1}]}) \quad \text{.............................. (26)}$$

Nonlinear regression can be applied to Eq. (26) for the simultaneous estimation of reservoir original hydrocarbons in place N and G_i in addition to the aquifer parameters B, c and R_{eD} . However, the objective of this work is to test the procedure of performing regression analysis using the numerical inversion of Laplace transform of the dimensionless water influx functions, Therefor, we limited this work to the estimation of the aquifer parameters from We data.

The water influx We can be calculated from production and PVT data using Eq. (1) if accurate estimates of N and G_i are available. Also, We can be calculated if an estimate of the volume of the invaded zone, V_{Pinv} , is obtained from the movement of the water/oil contact and structural and porosity maps of the reservoir using the equation

$$We = W_P\beta_w + V_{Pinv}\phi(1 - S_{wi} - S_{or}) \quad \text{.................... (27)}$$

Given values of water influx We and average pressure drop in the reservoir ΔP at different times, the method of least squares is applied to estimate aquifer parameters. The method is used to find the values of the three aquifer parameters B , C and R_{eD} which minimize the sum of squares of the differences between observed water influx values and those calculated using the three parameters for all data points. So it is required to minimize the following objective function

$$SSE = \sum_{k=1}^{Nt}\left[We_k - B\sum_{j=1}^{k}\Delta m_j \frac{\ln 2}{c^2[t_k - t_{j-1}]}\sum_{i=1}^{N} V_i\widetilde{\overline{Q}}(\frac{i\ln 2}{c[t_k - t_{j-1}]})\right]^2 \quad \text{......................... (28)}$$

In this equation the slope m is evaluated w.r.t the real time t which is c times the slope w.r.t the dimensionless time t_D.This equation can be written as

$$SSE(B, c, R_{eD}) = \sum_{k=1}^{Nt} Rs_k^2 \quad \text{.............................. (29)}$$

where the residual (error) Rs is given by

$$Rs_k = We_k - B\sum_{j=1}^{k}\Delta m_j \frac{\ln 2}{c^2[t_k - t_{j-1}]}\sum_{i=1}^{N} V_i\widetilde{\overline{Q}}(\frac{i\ln 2}{c[t_k - t_{j-1}]} \quad \text{................................ (30)}$$

The minamility conditions for the objective function are:

$$g_i = \frac{\partial SSE}{\partial \theta_i} = 2\sum_{k=1}^{Nt}[Rs\frac{\partial Rs}{\partial \theta_i}]_k = 0 \quad \text{.................. (31)}$$

Where θ is the vector of parameters B, c, and R_{eD} The set of Equations (31) are the normal equations of the system and must be solved simultaneously for parameters θ. Since the equations are nonlinear, an iterative procedure must be used. The application of the multi-variable Newton Rapson method to the normal equations results in the well known Gauss method of optimization. The solution at any iterative step is given in a matrix form as

$$H\delta\theta = -g \quad \text{.. (32)}$$

Where H_{ij}'s are the elements of the Hessian matrix which are the partial derivatives of the normal functions g_i with respect to parameter θ_j and the vector $\delta\theta$ is the changes in the values of the parameters in the iteration step. From Eq (31),

$$H_{ij} = \frac{\partial g_i}{\partial \theta_j} = 2\sum_{k=1}^{Nt}[\frac{\partial Rs}{\partial \theta_i}\frac{\partial Rs}{\partial \theta_j} + Rs\frac{\partial^2 Rs}{\partial \theta_i \partial \theta_j}]_k \quad \text{... (33)}$$

Expressions for the elements of the matrix H and vector g are derived in the Appendix.

The Gauss method uses the two terms in Eq.(33) for the Hessian matrix and thus requires evaluation of second derivatives. A modified form which does not require evaluation of the second derivatives is the Gauss-Newton method. In this method, only the first term is used which involves products of first derivatives only. The justification for dropping the second term in the Hessian matrix is that the second derivatives in that term are multiplied by the residual Rs which is supposed to be very small close to the solution.

This however might not be the case at points far from the solution. In this work we will consider the full form of the Hessian matrix since second derivatives can be evaluated analytically when the Stehfest algorithm is used for the numerical inversion of Laplace transform as shown in the appendix.

Although it is known that the Gauss method and the modified Gauss-Newton method have quadratic convergence near the solution, it may diverge if the initial guess is far from the solution. On the other hand, the method of steepest descent guarantees decreasing the objective function in each iteration but its convergence is slow. Methods presented by Levenberg and Marquardt are combinations of Gauss and the steepest descent methods. To gurantee that the objective function is decreasing, the Hessian matrix H must be positive definite. It can be made so by adding large positive numbers to its diagonal elements. Thus, the matrix H in Eq. (32) is replaced by $H+\lambda V$ where λ is a positive number and V is a diagonal matrix. Levenberg[14] took V as the identity matrix I while a better choice suggested by Marquardt[15] is to take the diagonal elements of V equal to the absolute values of the diagonal elements of H. So, the system of equations to be solved at each iteration becomes

$$[H+\lambda V]\delta\theta = -g \quad \text{.................................} \quad (34)$$

A very large value of λ $(\lambda\to\infty)$ is equivalent to the steepest descent method while λ=0 represents to the Gauss method.

The system of the three linear equations represented by Eq. (34) can be solved by the Gauss elimination method or by Cramer's rule. The iteration is continued until a convergence criterion is achieved. Either the sum of squares of errors SSE or $||\ g||$, the norm of the vector g, is used.

$$[[g]] = \sqrt{g_1^2 + g_2^2 + g_3^2} \quad \text{..............................} \quad (35)$$

Investigation of Eq. (29)-(31) shows that both expressions include residuals of the water influx terms at the data points, i.e. the differences between observed and calculated values. Since water influx values run into millions of barrels, the magnitude of SSE or , $||\ g||$ would be in the order of 1.0 E+8 for a 1% relative difference of the residuals. To overcome this problem, the residuals Rs_k are normalized by dividing each term by the water influx at that point, We_k. A relative difference of 1% in the residals will result in a sum of squares of 1.0E-4 multiplied by the number of data points. A convergence criterion of SSE = 1.0E-5 amounts to a relative error of 0.1% for a run of 10 data points.

Results and Discussion

The data generated for constant water influx rate is used to test the proposed method. A computer program is written using the developed procedure. The minimization wea performed on the normalized sum of squares of residuals. The expression for Rs_k in Eq. (30) was divided by We_k. The assumed values for the aquifer parameters R_{eD} , B, and c were 5, 22.38 res. Bbl/psi, and 0.6336 day^{-1} respectively.

To test the performance of the algorithm, two values for each parameter were used, one too small and the other too large compared to the actual value. All 8 combinations of the values were used as starting points. The results of these runs are shown in Table 3. It is seen from the results that all runs converged to values very close to the actual values (less than 1.5% for c and 0.5% for both b and R_{eD}).

Table 4 shows the results for one of the runs. It is seen that the absolute percentage error in the values of the water influx We ranges between a minimum value of about 0.0002% to a maximum value of about 0.32 % . With the value of SEE of 1.88 E-5 for 13 data points, the average absolute relative error is 0.12%. The values of transmissibility T and storativity S calculated from B and c in Table 4 using Eq. (12) and (13) are 1986.4 md.ft/cp and 2.0102 x 10^{-5} ft/psi as compared to the actual values of 2000 md.ft/cp and 2.0 x 10^{-5} ft/psi respectively. The error is 0.7% in T and 0.5% in S.

Results of table 3 show that the method converges always to the correct values of aquifer parameters regardless of the values at initial guess. No initial guess converged to a different solution and so a unique solution was obtained. The initial guess points were choosen to cover a wide range of parameter values and represent the corners of a rectangular parallelepiped around the correct value. It is therefore expected that if a fairly reasonable estimate of the aquifer parameters can be made (from 0.2 to 5 of the correct values), the procedure would converge to the correct values. The aquifer constant B depends on the thickness h, the porosity ϕ and the aquifer radius r_w. Both h and ϕ can be reasonably estimated. A value for r_w.between 50% and 200% of the tru value can be estimated from geological and geophysical data with reasonable certainity. The time adjustment factor c depends on the permeability K, the viscosity μ, the porosity ϕ, and the compressibility C. All these properties can also be estimated with reasonable accuracy between 50% and 200% of actual values. The same can also be stated for the dimensionless aquifer radius R_{eD} . If the initial guess was made within these limits, the procedure is then expected to converge to the neighborhood of the true value.

Table 3: Results of Regression Analysis Runs

Run	Initial Guess			Final Results			SSE
No.	R_{eD}	B	C	R_{eD}	B	C	
1	2	5	0.1	4.9823	22.4991	0.6261	1.8687E5
2	2	5	5	4.9855	22.4647	0.6274	1.8708E5
3	2	200	0.1	4.9825	22.4943	0.6261	1.8656E5
4	2	200	5	4.9823	22.4964	0.6260	1.8824E5
5	10	5	0.1	4.9823	22.4964	0.6260	1.8656E5
6	10	5	5	4.9830	22.4901	0.6263	1.8813E5
7	10	200	0.1	4.9816	22.5042	0.6257	1.8746E5
8	10	200	5	4.9830	22.4899	0.6263	1.8738E5

Table 4: Results of Regression Analysis

	Values of Parameters	
	Regression	(Actual)
R_{eD} =	4.982437	(5)
B =	22.49596	(22.38)
c =	0.6261127	(0.6333)

SSE **= 1.87667E-5**

|| *g* || = 2.69344E-9

Time (days)	We Observed	We Calculated	% Abs Error
1	794.72	794.90	0.02273
2	1851.40	1851.03	0.02008
5	4829.29	4829.28	0.00017
10	9812.74	9809.32	0.03490
20	19858.75	19865.22	0.03257
30	29948.21	29954.64	0.02146
40	40020.18	40004.84	0.03834
50	50021.96	50017.62	0.00868
60	60077.60	60022.37	0.09194
70	69924.31	70011.60	0.12483
80	79736.34	79990.59	0.31885
90	90173.63	89991.76	0.20169
100	100117.80	99990.84	0.12680

Conclusions

The following conclusions can be made

1- A method is presented for aquifer parameter estimations by the method of least squares applied to unsteady state van Everdingen and Hurst solution in Laplase domain. The parameters estimated are the relative aquifer size, the aquifer constant *B* and the time adjustment factor *c*. The parameters *B* and *c* can be used to calculate the transmissibility *T* and storativity *S* of the aquifer.

2- The Stehfest algorithm for numerical inversion of Laplase Transform was used to evaluate the water influx and obtain analytical expressions for the first and second derivatives of the solution with respect to aquifer parameters.

3- The Levenberg-Marquardt method was used in parameter estimation to avoid divergence for initial guesses not close to the solution . The method achieved convergence to a unique solution in the neighborhood of the real parameters for widely separated starting points. No divergence or convergence to different solutions was observed.

4- The pressure history is approximated by a series of linear segments (LP method) rather than the stair-like pressure steps (SP method). The Lp method was foud to give more accurate results.

5- The developed procedure can be applied for simultaneous estimation of original hydrocarbon in place OHIP (oil ab\nd/or gas) and aquifer parameters.

Nomenclature

B = aquifer constant, bbl/psi [m^3/kPa]
c = time adjustment factor, day^{-1}[s^{-1}]
C_t = total formation compressibility, psi^{-1} [kPa^{-1}]
G_i = initial gas in place, SCF [st m^3]
G_P = cumulative gas production, SCF [st m^3]
H = Hessian matrix
I_l = modified Bessel function of first kind, order l
J_l = Bessel function of first kind, order l
K_l = modified Bessel function of second kind, order l
K = absolute permeability, md [μm^2]
m = slope of pressure, psi/day [kPa/s]
N = initial oil in place, STB [st m^3]
N = Stehfest algorithm number
Np = cumulative oil production STB [m^3]
N_t = total number of data points
ΔP = pressure drop, psi [kPa]
q = flow rate, bbl/d [m^3/s]
$Q(t_D)$ = Dimensionless water influx
$\overline{Q}(s)$ = Laplace transform of $Q(t_D)$
r_w = reservoir radius, ft [m]
R_{eD} = dimensionless aquifer radius
Rs = gas soulibility in oil, SCF/STB
s = Laplase transform parameter
S = storativity $h\phi C$, ft.psi^{-1}[m kPa^{-1}]
S_{wi} = initial water saturation, fraction
t = time, d [s]
t_D = dimensionless time
T = transmissibility Kh/μ , md.ft/cp [μm^3/Pa.s]
We = water influx, Bbl [m^3]
Wp = water production, Bbl [m^3]
Y_l = Bessel function of second kind, order l
β_O = oil formation volume factor, res. Bbl/STB
β_g = gas formation volume factor, res. Ft^3/SCF
μ = viscosity, cp.[Pa.s]
λ = Levenberg parameter
φ = porosity, fraction

Subscripts

g = gas
i = initial
o = oil
w = water

Superscripts

— = Laplace transform

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Appendix – Derivation of the Elements of the Hessian Matrix

The sum of squares of errors to be minimized is

$$SSE = \sum_{k=1}^{Nt}\left[We_k - B\sum_{j=1}^{k}\Delta m_j \frac{\ln 2}{c^2[t_k - t_{j-1}]}\sum_{i=1}^{N} V_i \bar{\bar{Q}}\left(\frac{i\ln 2}{c[t_k - t_{j-1}]}\right)\right]^2 \quad \text{……………………..(A-1)}$$

.Which may be written as

$$SEE(B,c,R_{eD}) = \sum_{k=1}^{Nt} Rs_k^2 \quad \text{……………… . (A-2)}$$

where the residual (error) Rs is given by

$$Rs_k = We_k - B\sum_{j=1}^{k}\Delta m_j \frac{\ln 2}{c^2[t_k - t_{j-1}]}\sum_{i=1}^{N} V_i \bar{\bar{Q}}\left(\frac{i\ln 2}{c[t_k - t_{j-1}]}\right) \quad \text{…………………………… (A-3)}$$

From Eq. (A-3), the derivatives of the residual Rs_k w,r.t. the parameters B, c and R_{eD} are given by

$$\frac{\partial Rs_k}{\partial B} = -\sum_{j=1}^{k}\Delta m_j \frac{\ln 2}{c^2[t_k - t_{j-1}]}\sum_{i=1}^{N} V_i \bar{\bar{Q}}\left(\frac{i\ln 2}{c[t_k - t_{j-1}]}\right) \quad \text{….. (A-4)}$$

$$\frac{\partial Rs_k}{\partial R_{eD}} = -B\sum_{j=1}^{k}\Delta m_j \frac{\ln 2}{c^2[t_k - t_{j-1}]}\sum_{i=1}^{N} V_i \frac{\partial}{\partial R_{eD}}\left[\bar{\bar{Q}}\left(\frac{i\ln 2}{c[t_k - t_{j-1}]}\right)\right] \quad \text{..(A-5)}$$

$$\frac{\partial Rs_k}{\partial c} = -B\sum_{j=1}^{k}\Delta m_j \frac{\ln 2}{[t_k - t_{j-1}]}\left[\begin{array}{l}\sum_{i=1}^{N}\frac{V_i}{c^2}\left\{\frac{\partial}{\partial c}\left[\bar{\bar{Q}}\left(\frac{i\ln 2}{c[t_k - t_{j-1}]}\right)\right]\right. \\ \left. -\frac{2}{c}\bar{\bar{Q}}\left(\frac{i\ln 2}{c[t_k - t_{j-1}]}\right)\right\}\end{array}\right] \quad \text{………………….. (A-6)}$$

The secobd derivatives of the residuals are obtained by differentiation of Eq. (A-4) - (A-6).

$$\frac{\partial^2 Rs_k}{\partial B^2} = 0 \quad \text{…………………………………………... (A-7)}$$

$$\frac{\partial^2 Rs_k}{\partial R_{eD}\partial B} = -\sum_{j=1}^{k}\Delta m_j \frac{\ln 2}{c^2[t_k - t_{j-1}]}\sum_{i=1}^{N} V_i \frac{\partial}{\partial R_{eD\,i}}\left[\bar{\bar{Q}}\left(\frac{i\ln 2}{c[t_k - t_{j-1}]}\right)\right] \quad \text{….(A-8)}$$

$$\frac{\partial^2 Rs_k}{\partial c\partial B} = -\sum_{j=1}^{k}\Delta m_j \frac{\ln 2}{[t_k - t_{j-1}]}\left[\sum_{i=1}^{N}\frac{V_i}{c^2}\left\{\frac{\partial}{\partial c}\bar{\bar{Q}} - \frac{2}{c}\bar{\bar{Q}}\right\}\right] \quad \text{…....(A-9)}$$

$$\frac{\partial^2 Rs_k}{\partial c\partial R_{eD}} = -B\sum_{j=1}^{k}\Delta m_j \frac{\ln 2}{[t_k - t_{j-1}]}\left[\sum_{i=1}^{N}\frac{V_i}{c^2}\left\{\frac{\partial^2 \bar{\bar{Q}}}{\partial c\partial R_{eD}} - \frac{2}{c}\frac{\partial}{\partial R_{eD}}\bar{\bar{Q}}\right\}\right] \quad \text{..(A-10)}$$

$$\frac{\partial^2 Rs_k}{\partial c^2} = -B\sum_{j=1}^{k}\Delta m_j \frac{\ln 2}{[t_k - t_{j-1}]}\left[\sum_{i=1}^{N}\frac{V_i}{c^2}\left\{\frac{\partial^2}{\partial c^2}\bar{\bar{Q}} - \frac{4}{c}\frac{\partial}{\partial c}\bar{\bar{Q}} + \frac{6}{c^2}\bar{\bar{Q}}\right\}\right] \quad \text{………………….. (A-11)}$$

$$\frac{\partial^2 Rs_k}{\partial R^2_{eD}} = -B\sum_{j=1}^{k}\Delta m_j \frac{\ln 2}{c^2[t_k - t_{j-1}]}\sum_{i=1}^{N} V_i \frac{\partial^2}{\partial R^2_{eD}}\left[\bar{\bar{Q}}\left(\frac{i\ln 2}{c[t_k - t_{j-1}]}\right)\right] \quad \text{(A-12)}$$

From Eq. (10) and (16)

$$\bar{\bar{Q}}(s) = \frac{I_1(\sqrt{s}R_{eD})K_1(\sqrt{s}) - K_1(\sqrt{s}R_{eD})I_1(\sqrt{s})}{s^{5/2}\left[K_1(\sqrt{s}R_{eD})I_0(\sqrt{s}) + I_1(\sqrt{s}R_{eD})K_0(\sqrt{s})\right]} \quad \text{…………………. (A-13)}$$

Using

$$s = \frac{i \ln 2}{c[t_k - t_{j-1}]} = \frac{z_{ijk}}{c} \quad \text{(A-14)}$$

$$\overline{\widetilde{Q}} = \frac{c^{5/2} A}{z_{ijk}^{5/2} D} \quad \text{(A-15)}$$

where

$$A = I_1\left(\sqrt{\frac{z_{ijk}}{c}} R_{eD}\right) K_1\left(\sqrt{\frac{z_{ijk}}{c}}\right) - K_1\left(\sqrt{\frac{z_{ijk}}{c}} R_{eD}\right) I_1\left(\sqrt{\frac{z_{ijk}}{c}}\right) \quad \text{(A-16)}$$

$$D = K_1\left(\sqrt{\frac{z_{ijk}}{c}} R_{eD}\right) I_0\left(\sqrt{\frac{z_{ijk}}{c}}\right) + I_1\left(\sqrt{\frac{z_{ijk}}{c}} R_{eD}\right) K_0\left(\sqrt{\frac{z_{ijk}}{c}}\right) \quad \text{(A-17)}$$

The partial derivatives of $\overline{\widetilde{Q}}$ w.r.t. c and R_{eD} are given by

$$\frac{\partial \overline{\widetilde{Q}}}{\partial R_{eD}} = \frac{c^{5/2}}{z_{ijk}^{5/2}} \left(\frac{DA_r - AD_r}{D^2} \right) \quad \text{(A-18)}$$

$$\frac{\partial \overline{\widetilde{Q}}}{\partial c} = \frac{c^{5/2}}{z_{ijk}^{5/2}} \left(\frac{DA_c - AD_c}{D^2} \right) + \frac{5 \overline{\widetilde{Q}}}{2c} \quad \text{(A-19)}$$

$$\frac{\partial^2 \overline{\widetilde{Q}}}{\partial R^2_{eD}} = \frac{c^{5/2}}{z_{ijk}^{5/2}} \left(\frac{DA_{rr} - 2D_r A_r - AD_{rr}}{D^2} + \frac{2AD}{D^3} \right) \quad \text{(A-20)}$$

$$\frac{\partial^2 \overline{\widetilde{Q}}}{\partial c^2} = \frac{5c^{3/2}}{2 z_{ijk}^{5/2}} \left(\frac{DA_c - AD_c}{D^2} \right) - \frac{5 \overline{\widetilde{Q}}}{2c^2} + \frac{5}{2c} \frac{\partial \overline{\widetilde{Q}}}{\partial c} + \frac{c^{5/2}}{z_{ijk}^{5/2}} \left(\frac{DA_{cc} - 2D_c A_c - AD_{cc}}{D^2} + \frac{2AD_c^2}{D^3} \right) \quad \text{(A-21)}$$

$$\frac{\partial^2 \overline{\widetilde{Q}}}{\partial c \partial R_{eD}} = \frac{c^{5/2}}{z_{ijk}^{5/2}} \left(\begin{array}{l} \frac{A_{cr}}{D} + \frac{2AD_r D_c}{D^3} \\ - \frac{AD_{cr} + A_r D_c + A_c D_r}{D^2} \end{array} \right) + \frac{5}{2c} \frac{\partial \overline{\widetilde{Q}}}{\partial R_{eD}} \quad \text{(A-22)}$$

The partial derivatives of A and D with respect to R_{eD} and c are

$$A_r = \sqrt{\frac{z_{ijk}}{c}} \left[I_1'\left(\sqrt{\frac{z_{ijk}}{c}} R_{eD}\right) K_1\left(\sqrt{\frac{z_{ijk}}{c}}\right) - K_1'\left(\sqrt{\frac{z_{ijk}}{c}} R_{eD}\right) I_1\left(\sqrt{\frac{z_{ijk}}{c}}\right) \right] \quad \text{(A-23)}$$

$$D_r = \sqrt{\frac{z_{ijk}}{c}} \left[K_1'\left(\sqrt{\frac{z_{ijk}}{c}} R_{eD}\right) I_0\left(\sqrt{\frac{z_{ijk}}{c}}\right) + I_1'\left(\sqrt{\frac{z_{ijk}}{c}} R_{eD}\right) K_0\left(\sqrt{\frac{z_{ijk}}{c}}\right) \right] \quad \text{(A-24)}$$

$$A_c = -\sqrt{\frac{z_{ijk}}{c^3}} \left[\begin{array}{l} R_{eD}[I_1'(\sqrt{\frac{z_{ijk}}{c}} R_{eD}) K_1(\sqrt{\frac{z_{ijk}}{c}}) - K_1'(\sqrt{\frac{z_{ijk}}{c}} R_{eD}) I_1(\sqrt{\frac{z_{ijk}}{c}})] \\ + I_1(\sqrt{\frac{z_{ijk}}{c}} R_{eD}) K_1'(\sqrt{\frac{z_{ijk}}{c}}) - K_1(\sqrt{\frac{z_{ijk}}{c}} R_{eD}) I_1'(\sqrt{\frac{z_{ijk}}{c}}) \end{array} \right] \quad \text{(A-25)}$$

.

$$D_c = -\sqrt{\frac{z_{ijk}}{c^3}} \left[\begin{array}{l} R_{eD}[K_1'(\sqrt{\frac{z_{ijk}}{c}} R_{eD}) I_0(\sqrt{\frac{z_{ijk}}{c}}) + I_1'(\sqrt{\frac{z_{ijk}}{c}} R_{eD}) K_0(\sqrt{\frac{z_{ijk}}{c}})] \\ + K_1(\sqrt{\frac{z_{ijk}}{c}} R_{eD}) I_0'(\sqrt{\frac{z_{ijk}}{c}}) + I_1(\sqrt{\frac{z_{ijk}}{c}} R_{eD}) K_0'(\sqrt{\frac{z_{ijk}}{c}}) \end{array} \right] \quad \text{(A-26)}$$

The second partial derivatives of A and D with respect to R_{eD} and c are

$$A_{rr} = \frac{z_{ijk}}{c} \left[I_1''\left(\sqrt{\frac{z_{ijk}}{c}} R_{eD}\right) K_1\left(\sqrt{\frac{z_{ijk}}{c}}\right) - K_1''\left(\sqrt{\frac{z_{ijk}}{c}} R_{eD}\right) I_1\left(\sqrt{\frac{z_{ijk}}{c}}\right) \right] \quad \text{(A-27)}$$

$$D_{rr} = \frac{z_{ijk}}{c} \left[K_1''\left(\sqrt{\frac{z_{ijk}}{c}} R_{eD}\right) I_0\left(\sqrt{\frac{z_{ijk}}{c}}\right) + I_1''\left(\sqrt{\frac{z_{ijk}}{c}} R_{eD}\right) K_0\left(\sqrt{\frac{z_{ijk}}{c}}\right) \right] \quad \text{(A-28)}$$

$$A_{cc} = \frac{-3A_c}{2c} - \frac{z_{ijk}}{c^3} \left[\begin{array}{l} R_{eD}^2[I_1''(\sqrt{\frac{z_{ijk}}{c}} R_{eD}) K_1(\sqrt{\frac{z_{ijk}}{c}}) - K_1''(\sqrt{\frac{z_{ijk}}{c}} R_{eD}) I_1(\sqrt{\frac{z_{ijk}}{c}})] \\ + 2R_{eD}[I_1'(\sqrt{\frac{z_{ijk}}{c}} R_{eD}) K_1'(\sqrt{\frac{z_{ijk}}{c}}) - K_1'(\sqrt{\frac{z_{ijk}}{c}} R_{eD}) I_1'(\sqrt{\frac{z_{ijk}}{c}})] \\ + I_1(\sqrt{\frac{z_{ijk}}{c}} R_{eD}) K_1''(\sqrt{\frac{z_{ijk}}{c}}) - K_1(\sqrt{\frac{z_{ijk}}{c}} R_{eD}) I_1''(\sqrt{\frac{z_{ijk}}{c}}) \end{array} \right] \quad \text{(A-29)}$$

.

$$D_{cc} = \frac{-3D_c}{2c} - \frac{z_{ijk}}{c^3} \left[\begin{array}{l} R_{eD}^2[K_1''(\sqrt{\frac{z_{ijk}}{c}} R_{eD}) I_0(\sqrt{\frac{z_{ijk}}{c}}) + I_1''(\sqrt{\frac{z_{ijk}}{c}} R_{eD}) K_0(\sqrt{\frac{z_{ijk}}{c}})] \\ 2R_{eD}[K_1'(\sqrt{\frac{z_{ijk}}{c}} R_{eD}) I_0'(\sqrt{\frac{z_{ijk}}{c}}) + I_1'(\sqrt{\frac{z_{ijk}}{c}} R_{eD}) K_0'(\sqrt{\frac{z_{ijk}}{c}})] \\ + K_1(\sqrt{\frac{z_{ijk}}{c}} R_{eD}) I_0''(\sqrt{\frac{z_{ijk}}{c}}) + I_1(\sqrt{\frac{z_{ijk}}{c}} R_{eD}) K_0''(\sqrt{\frac{z_{ijk}}{c}}) \end{array} \right] \quad \text{(A-30)}$$

$$A_{rc}=-\sqrt{\frac{z_{ijk}}{c^3}}[I_1'(\sqrt{\frac{z_{ijk}}{c}}R_{eD})K_1(\sqrt{\frac{z_{ijk}}{c}})-K_1'(\sqrt{\frac{z_{ijk}}{c}}R_{eD})I_1(\sqrt{\frac{z_{ijk}}{c}})]$$
$$-\frac{z_{ijk}}{c^2}\begin{bmatrix} R_{eD}[I_1''(\sqrt{\frac{z_{ijk}}{c}}R_{eD})K_1(\sqrt{\frac{z_{ijk}}{c}})-K_1''(\sqrt{\frac{z_{ijk}}{c}}R_{eD})I_1(\sqrt{\frac{z_{ijk}}{c}})] \\ +I_1'(\sqrt{\frac{z_{ijk}}{c}}R_{eD})K_1'(\sqrt{\frac{z_{ijk}}{c}})-K_1'(\sqrt{\frac{z_{ijk}}{c}}R_{eD})I_1'(\sqrt{\frac{z_{ijk}}{c}}) \end{bmatrix}$$

……....………….. (A-31)

.

$$D_{rc}=-\sqrt{\frac{z_{ijk}}{c^3}}[K_1'(\sqrt{\frac{z_{ijk}}{c}}R_{eD})I_0(\sqrt{\frac{z_{ijk}}{c}})+I_1'(\sqrt{\frac{z_{ijk}}{c}}R_{eD})K_0(\sqrt{\frac{z_{ijk}}{c}})]$$
$$+\frac{z_{ijk}}{c^2}\begin{bmatrix} R_{eD}[K_1''(\sqrt{\frac{z_{ijk}}{c}}R_{eD})I_0(\sqrt{\frac{z_{ijk}}{c}})+I_1''(\sqrt{\frac{z_{ijk}}{c}}R_{eD})K_0(\sqrt{\frac{z_{ijk}}{c}})] \\ +K_1'(\sqrt{\frac{z_{ijk}}{c}}R_{eD})I_0'(\sqrt{\frac{z_{ijk}}{c}})+I_1'(\sqrt{\frac{z_{ijk}}{c}}R_{eD})K_0'(\sqrt{\frac{z_{ijk}}{c}}) \end{bmatrix}$$

…..……....…….. (A-32)

Where the derivatives of the Bessel functions are

$$I_0'(x)=I_1(x) \quad \text{……….………………....………….. (A-33)}$$

$$K_0'(x)=-K_1(x) \quad \text{…….………………..…...……….. (A-34)}$$

$$I_1'(x)=I_0(x)-\frac{I_1(x)}{x} \quad \text{………………..…….…….. (A-35)}$$

$$K_1'(x)=-K_0(x)-\frac{K_1(x)}{x} \quad \text{………..……...……….. (A-36)}$$

$$I_1''(x)=I_1(x)-\frac{I_0(x)}{x}+\frac{2I_1(x)}{x^2} \quad \text{….…….……….. (A-37)}$$

$$K_1''(x)=K_1(x)+\frac{K_0(x)}{x}+\frac{2K_1(x)}{x^2} \quad \text{…..……….... (A-38)}$$

SI Metric Conversion Factors

bbl	x	1.589 873	E – 01 = m^3
cp	x	1.0*	E – 03 = Pa.s
ft	x	3.048*	E – 01 = m
ft^2	x	9.290 304*	E – 02 = m^2

*Conversion factor is exact.