Waterflooding Performance of Communicating Stratified Reservoirs With Log-Normal Permeability Distribution

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Summary

An analytical solution is developed for waterflooding performance of layered reservoirs with a log-normal permeability distribution with complete crossflow between layers. The permeability distribution is characterized by the Dykstra-Parsons (DP) variation coefficient V_{DP} or the standard deviation of the distribution σ_k. The performance is expressed in terms of vertical coverage as function of the producing water-oil ratio. Also an expression for the dimensionless time (pore volumes of injected water) at a given water-oil ratio is derived. Expressions are also derived for pseudorelative permeability functions and fractional flow curves that can be used in reservoir simulation. Correlation charts are also presented to enable graphical determination of the performance. The variables are combined in such a way that a single chart is constructed for the entire range of water-oil ratio, mobility ratio and permeability variation.

Analogy to the Buckley-Leverett (BL) multiple-valued saturation profile is found to occur at low mobility ratios ($M<1$) where a multiple-valued displacement front is formed. A procedure similar to the BL discontinuity is suggested to handle this situation. Successive layers with different permeabilities are allowed to move with the same velocity resulting in a single-valued profile with a discontinuity. No such behavior is observed for mobility ratios greater than unity. A criterion for the minimum mobility ratio at which this behavior occurs is presented as a function of the variation coefficient V_{DP}.

Introduction

Waterflooding is still the recovery process responsible for most of the oil production by secondary recovery. Water injected into the reservoir displaces almost all of the oil except the residual oil saturation from the portions of the reservoir contacted or swept by water. The fraction of oil displaced from a contacted volume is known as the displacement efficiency and depends on the relative permeability characteristics of the rock as well as the viscosities of the displacing and displaced fluids. The extent to which a reservoir is swept by a displacing fluid is separated into areal and vertical sweep efficiencies. The areal sweep efficiency accounts for the nonlinearity of the flow patterns between injection and production wells. The vertical sweep efficiency or coverage is caused by the heterogeneity of the reservoir, i.e., variation of horizontal permeability in the vertical direction. The displacing fluid tends to move faster in zones with higher permeabilities, resulting in earlier breakthrough into producing wells. Both areal and vertical sweep efficiencies are highly dependent on the mobility ratio of the displacement process and depend on the volume of the injected fluid expressed in pore volumes. The vertical sweep efficiency, however, is mainly dependent on the permeability distribution in the producing layer. Because of the variation in the depositional environments, reservoir rocks usually exhibit random variations in their petrophysical properties. Porosity is usually found to have a normal distribution, while the permeability has a log-normal distribution. The log-normal distribution of permeability is characterized by two parameters: the mean permeability K_m and the standard deviation σ_k. The standard deviation σ_k can also be expressed in terms of the DP variation coefficient V_{DP}. It may also be related to the Lorenz coefficient L.

The methods available in the literature to predict the waterflooding performance of stratified reservoirs can be grouped into two categories depending on the assumption of communication or no communication between the different layers. The method of DP[1] is the basis for performance prediction in noncommunicating stratified reservoirs. In addition to the basic equations presented in their work, they also presented correlations of the vertical coverage for log-normal permeability distributions in terms of mobility ratio and permeability variation coefficient at different values of the water-oil ratio. Also presented in their paper is a correlation of actual recovery factor vs. vertical coverage, initial water saturation, and water-oil ratio. This correlation was based on experimental runs performed on core plugs with permeability distributions determined by measuring the permeability at different locations on the core with a minipermeameter. Johnson[2] later on combined the theoretical charts based on DP equations with the experimental correlation chart into a group of correlation charts from which the recovery factor at given values of water-oil ratio can be calculated directly without first computing the vertical coverage. Mobarek[3] found discrepancies between results obtained by this method and results obtained using a numerical model.

Muskat[4] presented analytical solution for waterflooding performance of stratified systems with linear and exponential permeability distributions. Reznik *et al.*[5] derived expressions for the variation of pressure drop or injection rate as function of injection time for the DP model.

Prediction of waterflooding performance for communicating reservoirs was presented by Hiatt.[6] This model assumes instantaneous crossflow between layers to keep the pressure gradient the same in all layers at any distance. Warren and Casgrove[7] applied the Hiatt model to a system with log-normal permeability distribution and normal porosity distribution. Their method is semigraphical, semianalytical since they obtain values from plots of permeability and formation capacity distributions on probability graphs. Hearn[8] used the same model of Hiatt to develop expressions for pseudorelative permeabilities that can be used in numerical reservoir simulation to reduce a three-dimensional model to a two-dimensional areal model with average (pseudo) functions for the vertical direction. El-Khatib[9] extended the work of Hiatt to account for variable rock properties other than the absolute permeability. He also presented equations for the variation of the injectivity ratio with injection time and compared performance of communicating and noncommunicating systems.

Since it is widely accepted that the permeability in reservoir rocks exhibits a log-normal distribution, the objective of this work is to present a solution in a closed form for the waterflooding performance of stratified reservoirs with such permeability distributions. This would be the limiting case for a stratified system composed of a very large number of layers. In such a case, it is reasonable to assume complete communication between the layers since it is highly unrealistic to assume such large number of layers to be separated by an equal number of thin insulating strata.

Assumptions and Definitions

The following assumptions are made:

- The system is linear, horizontal and of constant thickness.
- The flow is isothermal, incompressible and obeys Darcy's law.

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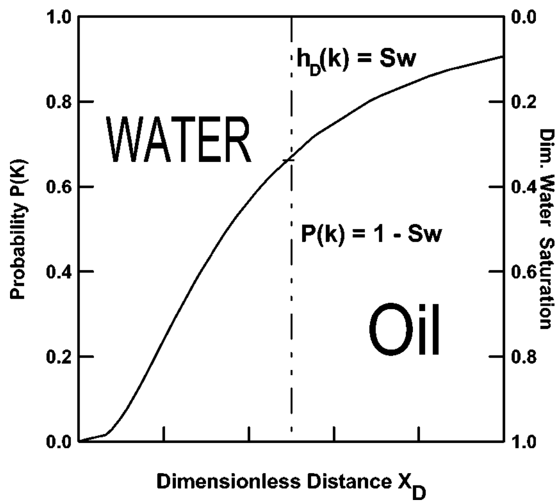

Fig. 1–Displacement front in the reservoir.

• The initial fluid distribution is uniform with irreducible water saturation.

• The flow is piston like with only oil flowing ahead of the displacement front and only water flowing behind it.

• Capillary and gravity forces are negligible.

• The relative permeability characteristics are the same for the entire system.

• The absolute permeability has a log-normal distribution

$$p(k) = 0.5 + 0.5 \,\text{erf}\left[\frac{\ln(k/k_m)}{\sqrt{2}\sigma_k}\right]. \tag{1}$$

• The porosity is assumed constant.

Although some of these assumptions limit the applicability of the model, the objective of the development is to estimate the vertical sweep efficiency (coverage) in the stratified system. To convert this into recovery factors, the coverage should be multiplied by the areal sweep efficiency of the flood pattern and by the displacement efficiency.

Fig. 1 is a schematic representation of the system. The following definitions are introduced to obtain a generalized form of the model:

h_D is the dimensionless cumulative thickness of section invaded by water,

$$h_D = 1 - P(k), \tag{2}$$

$P(k)$ is the probability of cumulative relative thickness of permeability less than or equal to k, S_D is the average dimensionless saturation at any cross section with thickness h_D invaded by water defined as

$$S_D = (S_w - S_{wi})/(1 - S_{wi} - S_{or}), \tag{3}$$

where S_w is the average saturation at the cross section,

$$S_w = S_{wi} + \Delta S h_D, \tag{4}$$

ΔS is the displaceable hydrocarbon saturation,

$$\Delta S = 1 - S_{wi} - S_{or}, \tag{5}$$

from which it follows that

$$S_D = h_D = 1 - P(k), \tag{6}$$

s is the dimensionless cumulative formation capacity

$$s = \frac{\int_0^{h_D} k \,\mathrm{d}z}{\int_0^1 k \,\mathrm{d}z} = \frac{\int_{p(k)}^1 k \,\mathrm{d}P(k)}{\int_0^1 k \,\mathrm{d}P(k)}, \tag{7}$$

$$F = 1 - s = \frac{\int_0^{p(k)} k \,\mathrm{d}P(k)}{\int_0^1 k \,\mathrm{d}P(k)}, \tag{8}$$

τ is the dimensionless time in pore volumes injected

$$\tau = \frac{5.62 \int_0^t q \,\mathrm{d}z}{A \phi L}, \tag{9}$$

$$M = \text{mobility ratio} = \frac{k_{rw}^0}{k_{ro}^0} \frac{\mu_0}{\mu_w}. \tag{10}$$

Development of the Model Equations

Following the derivation of Hiatt, Warren and Casgrove and El-Khatib the following equations are obtained:

$$f_w = Ms/[Ms + (1 - s)], \tag{11}$$

$$R = S_D + (1 - f_w)\tau, \tag{12}$$

$$\tau = 1/(\mathrm{d}f_w/\mathrm{d}S_D), \tag{13}$$

For the log-normal permeability distribution, the following equations for dimensionless time τ and vertical coverage R in terms of the water-oil ratio can be obtained:

$$\tau = \frac{[1 + F_{wo}]^2}{M(1+x)^2} \text{Exp}\left[-0.5\sigma_k^2 - \sqrt{2}\sigma_k \,\text{erf}^{-1}\left(\frac{1-x}{1+x}\right)\right], \tag{14}$$

and

$$R = 0.5 - 0.5 \,\text{erf}\left[\text{erf}^{-1}\left(\frac{1-x}{1+x}\right) + \frac{\sigma_k}{\sqrt{2}}\right] + \frac{\tau}{1 + F_{wo}}, \tag{15}$$

where

$$x = F_{wo}/M.$$

Eqs. 14 and 15 can be used to calculate τ and R at any value of water-oil ratio F_{wo} for a stratified system with standard deviation σ_k at a mobility ratio M. Derivation of these equations is given in Appendix A.

Computational Procedure

• Permeability data are tabulated in increasing order of K vs. relative cumulative thickness $P(K)$. The data are plotted on log-probability paper and the best straight line is drawn.

• The DP variation coefficient V_{DP} and hence the standard deviation σ_k are determined

$$V_{\text{DP}} = (k_{84.1} - k_{50})/k_{84.1}, \tag{16}$$

and

$$\sigma_k = \ln[1/(1 - V_{\text{DP}})]. \tag{17}$$

The data can also be fitted using the method of least squares as shown in Appendix B.

• For the specified value of the mobility ratio M and for increasing values of water oil ratio F_{wo}, Eq. 14 is used to calculate the dimensionless time τ.

• The value of τ obtained from step 3 is used in Eq. 15 to calculate the vertical coverage R. A computer program was written to fit the data and compute the performance of a given stratified system for a given mobility ratio M. **Fig. 2** shows the performance in terms of R vs. F_{wo} for different values of M at a value of V_{DP} of 0.5 and **Fig. 3** shows the performance for different values of V_{DP} at a mobility ratio of 2.

This procedure is applicable for permeability distributions that are log-normal. Although this distribution is widely accepted for permeability in oil reservoirs, situations may be present where the distribution cannot be adequately described by the log-normal curve. Such cases are easily recognized by the significant deviation of the data points from the best-fit straight line on the log-probability plot. Also if the number of layers with different permeabilities into which the reservoir is divided is not large, not enough data points will be available to construct the probability plot. Under these circumstances, the developed equations cannot be used to predict the performance of the system. The models

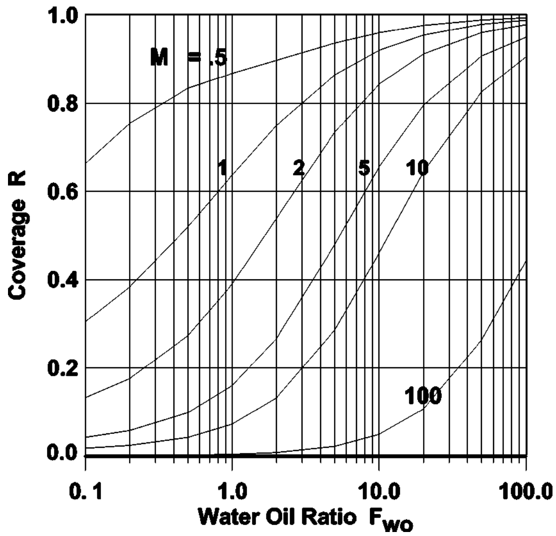

Fig. 2–Performance for $V_{DP}=0.5$.

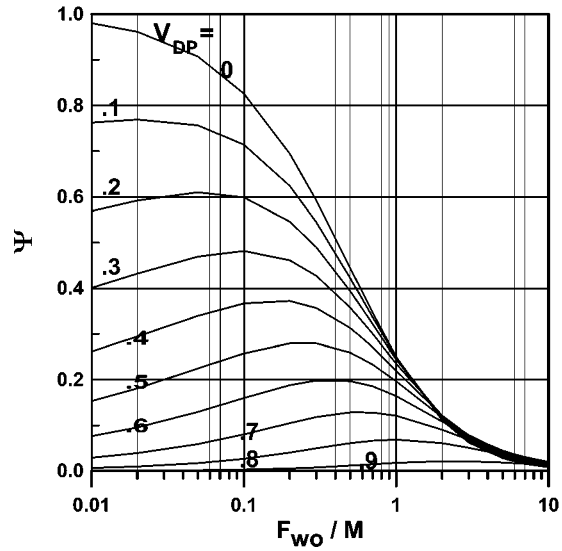

Fig. 4–Correlation chart for Ψ–normal scale.

developed for waterflooding performance of systems with arbitrary discrete permeability distributions as those presented by Hiatt *et al.* could be used.

Correlation Charts

Eqs. 14 and 15 indicate that the dependent variables of the performance τ and R are functions of the independent variable F_{wo} and the parameters M and σ_k (or V_{DP}). This requires the construction of several charts for each of the dependent variables.

Eqs. 14 and 15, however, can be rearranged to allow the construction of a single correlation chart for each of τ and R for the entire range of the independent variable and model parameters. These equations can be written in the following form:

$$\Psi(\sigma_k,x)=\frac{M\tau}{(1+F_{wo})^2}=\frac{1}{(1+x)^2}\,\mathrm{Exp}\left[-0.5\sigma_k^2-\sqrt{2}\sigma_k\,\mathrm{erf}^{-1}\left(\frac{1-x}{1+x}\right)\right], \quad (18)$$

and

$$\Phi(\sigma_k,x)=R-\frac{\tau}{1+F_w}=0.5-0.5\,\mathrm{erf}\left[\mathrm{erf}^{-1}\left(\frac{1-x}{1+x}\right)+\frac{\sigma_k}{\sqrt{2}}\right]. \quad (19)$$

Charts for Ψ and Φ vs. x for different values of σ_k (or V_{DP}) can be constructed. The values of τ and R are then calculated as follows:

$$\tau=(1+F_{wo})^2\Psi/M, \quad (20)$$

and

$$R=\Phi+\tau/(1+F_{wo}), \quad (21)$$

or

$$R=\Phi+[\Psi(1+F_{wo})]/M. \quad (22)$$

Charts for Ψ and Φ are shown in **Figs. 4 through 7**. It is found more convenient to present these charts in terms of the DP variation coefficient V_{DP} which varies between zero and unity rather than the standard deviation of the log-normal distribution of the permeability σ_k which varies between zero and infinity. V_{DP} and σ_k are related by Eqs. 16 and 17. The charts are presented with normal and logarithmic scales of the y axis. The logarithmic

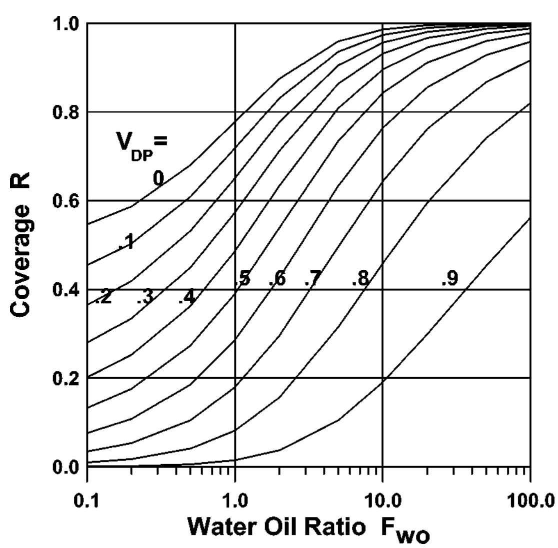

Fig. 3–Performance for $M=2$.

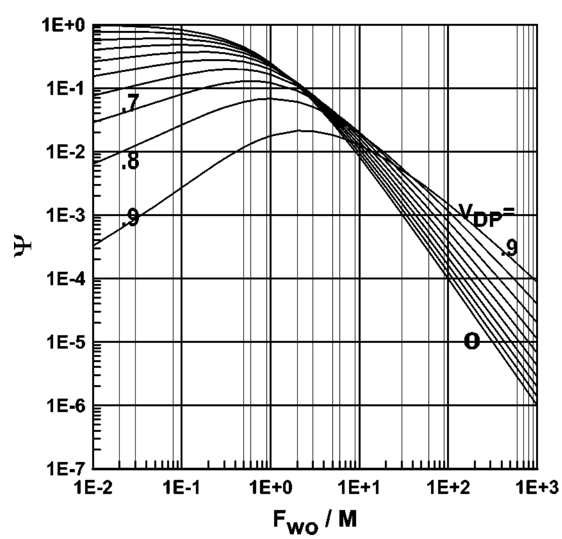

Fig. 5–Correlation chart for Ψ–log scale.

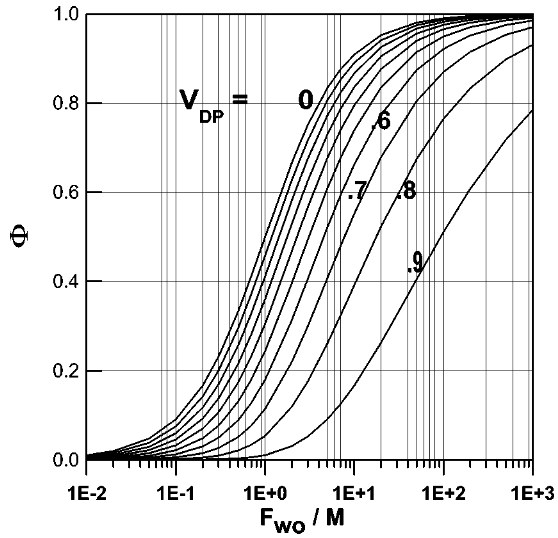

Fig. 6–Correlation chart for Φ–normal scale.

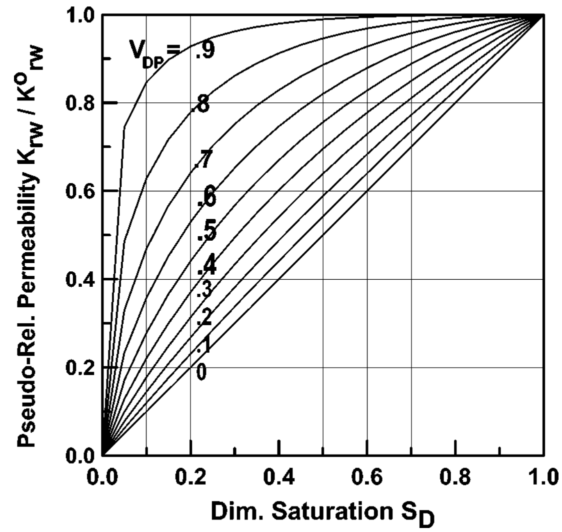

Fig. 8–Pseudorelative permeability curves.

scales can be used at low values of the dependent variables Ψ and τ, while the normal scales are used at values of these variables close to unity.

Pseudorelative Permeability Functions

As shown by Hearn, the pseudorelative permeability functions can be expressed as

$$\tilde{k}_{rw} = k_{rw}^0 s, \tag{23}$$

$$\tilde{k}_{ro} = k_{ro}^0 (1-s), \tag{24}$$

$$\tilde{S}_w = \Delta S \cdot (h/h_{tot}) + S_{wi}, \tag{25}$$

where k_{rw}^0 and k_{ro}^0 are the end points relative permeabilities.

In terms of the definitions used in this paper we can write

$$\tilde{k}_{rw}/k_{rw}^0 = s = 1-F, \tag{26}$$

$$\tilde{k}_{ro}/k_{ro}^0 = 1-s = F, \tag{27}$$

$$S_D = 1-P(k). \tag{28}$$

Substituting for $P(k)$ from Eq. 1, we get

$$S_D = 0.5 - 0.5\,\mathrm{erf}\left[\frac{\ln(k/k_m)}{\sqrt{2}\sigma_k}\right], \tag{29}$$

from which

$$\ln(k/k_m)/\sqrt{2}\sigma_k = \mathrm{erf}^{-1}(1/2S_D). \tag{30}$$

Substituting Eq. 30 into Eq. A-1 we get

$$F = 0.5 + 0.5\,\mathrm{erf}[\mathrm{erf}^{-1}(1-2S_D) - (\sigma_k/\sqrt{2})]. \tag{31}$$

Substituting for the value of F into Eqs. 26 and 27, we obtain

$$\tilde{k}_{rw}/k_{rw}^0 = 0.5 - 0.5\,\mathrm{erf}[\mathrm{erf}^{-1}(1-2S_D) - (\sigma_k/\sqrt{2})], \tag{32}$$

$$\tilde{k}_{ro}/k_{ro}^0 = 0.5 + 0.5\,\mathrm{erf}[\mathrm{erf}^{-1}(1-2S_D) - (\sigma_k\sqrt{2})]. \tag{33}$$

It is seen from Eqs. 32 and 33 that

$$\tilde{k}_{rw}/k_{rw}^0 + \tilde{k}_{ro}/k_{ro}^0 = 1. \tag{34}$$

It is therefore sufficient to calculate one from the two terms. Curves for the water pseudorelative permeability $\tilde{k}_{rw}$ are shown in **Fig. 8** for different values of V_{DP}. These curves can be used in reservoir simulation collapsing the vertical direction into a single block.

The pseudofractional flow curve f_w can be calculated from the relation

$$f_w = \frac{1}{1 + \dfrac{\tilde{k}_{ro}}{\tilde{k}_{rw}}\dfrac{\mu_w}{\mu_0}}. \tag{35}$$

Substituting Eqs. 32 and 33 for the pseudorelative permeabilities, we get

$$f_w = \frac{M\left\{1 - \mathrm{erf}\left[\mathrm{erf}^{-1}(1-2S_D) - \dfrac{\sigma_k}{\sqrt{2}}\right]\right\}}{(1+M) + (1-M)\,\mathrm{erf}\left[\mathrm{erf}^{-1}(1-2S_D) - \dfrac{\sigma_k}{\sqrt{2}}\right]}. \tag{36}$$

This can also be obtained by substituting the value of F from Eq. 31 into Eq. 11 with $s = 1-F$. Curves from the fractional flow vs. S_D are functions of the standard deviation σ_k (or V_{DP}) and mobility ratio M. Such curves can be used to predict the performance by the Welge[10] graphical techniques of the BL method. Curves for f_w vs. S_D, for mobility ratios of 0.2 and 2 for different values of V_{DP}, are shown in **Figs. 9 and 10**, respectively.

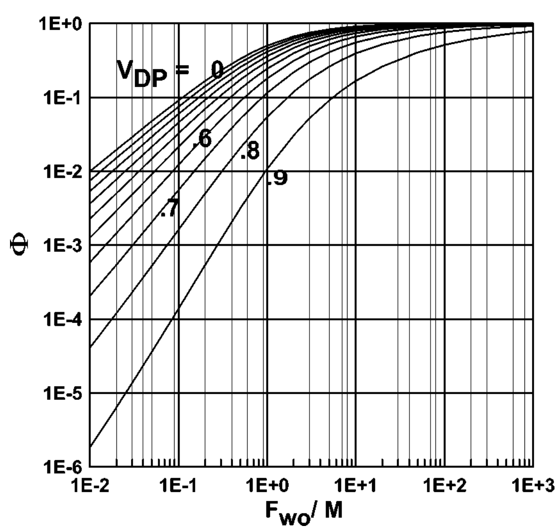

Fig. 7–Correlation chart for Φ–log. Scale.

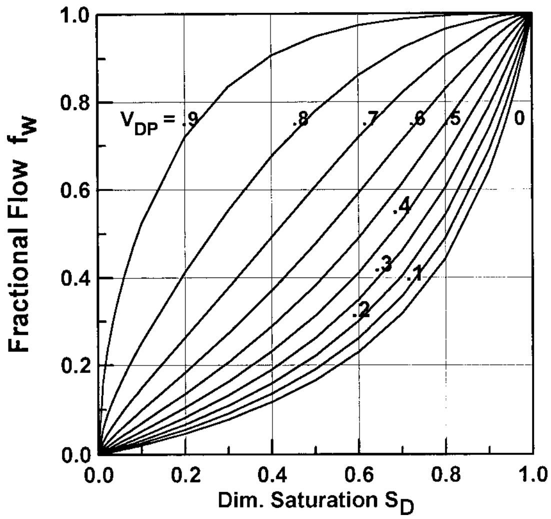

Fig. 9–Fractional flow curves, *M*=0.2.

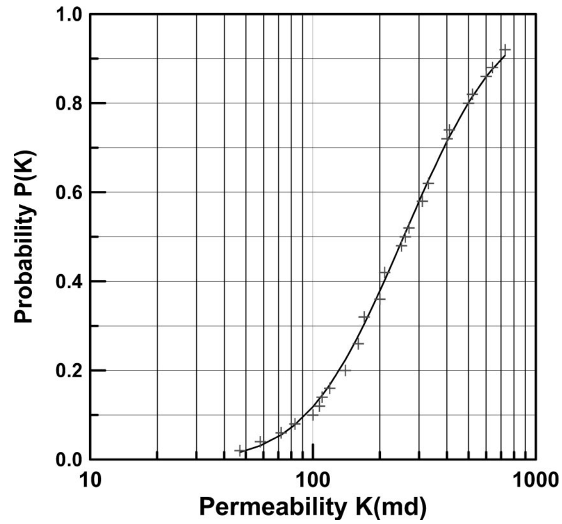

Fig. 11–Fitting of permeability distribution, normal graph.

Displacement at Low Mobility Ratio

A problem may arise in performance prediction of communicating satisfied systems at a low mobility ratio ($M<1$). The dimensionless time may not be continuously increasing as the water-oil ratio increases and vertical coverage may exceed unity at an intermediate stage during the displacement history. This problem was recognized by Hearn and El-Khatib but was ignored by others. It is clear from Eqs. 12 and 13 that the performance of the communicating stratified system is similar to that predicted by the BL procedure for a uniform single layer with average properties. In the original BL[11] work, it was observed that the rock relative permeability data lead to a multiple-valued saturation distribution (S_w vs. X). A discontinuity in saturation (shock) that satisfies the material balance concept was introduced to handle this situation. For the displacement in stratified systems, in order for the saturation distribution to be single valued, the distance of the front X must be a continuously increasing function of $P(k)$, i.e., continuously decreasing with S_D, h_D or s. Since the distance X is given by the equation

$$X=\tau(\mathrm{d}f_w/\mathrm{d}S_D). \tag{37}$$

The condition for a single valued distribution is

$$\mathrm{d}X_D/\mathrm{d}P(k)=-(\mathrm{d}X_D/\mathrm{d}S_D)\geqslant 0. \tag{38}$$

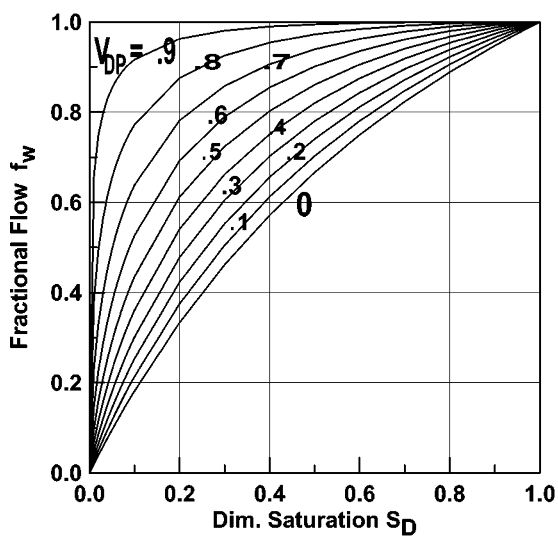

Fig. 10–Fractional flow curves, *M*=2.

It can be shown by differentiating Eq. A-10 w.r.t. $P(k)$ that the condition for a single valued saturation profile is stratified if

$$\sigma_k>\sqrt{\frac{2}{\pi}}\left(\frac{1-M}{M}\right), \tag{39}$$

or

$$M>\frac{1}{1+\sqrt{\pi/2}\sigma_k}. \tag{40}$$

However these conditions are sufficient but not necessary, i.e., a single-valued profile may occur at values of σ_k or M lower than those given by these equations.

For a reservoir with log-normal permeability distribution with a variation coefficient V_{DP}, if the mobility ratio M is less than that given by Eq. 40 a multiple-valued profile may be formed which is physically meaningless. In this case a procedure similar to that of BL must be used to obtain a single-valued profile. To satisfy the material balance concept, a shock (discontinuity) in the saturation is introduced. This is found by drawing a line tangential to the f_w–S_D curve at two points (S_1, f_{w1}), (S_2, f_{w2}) such that

$$(\mathrm{d}f_w/\mathrm{d}S_D)_1=(\mathrm{d}f_w/\mathrm{d}S_D)_2=(f_{w2}-f_{w1})/(S_{D2}-S_{D1}). \tag{41}$$

When this discontinuity reaches the outlet face (producing well), a jump occurs from S_{w1} to S_{w2} raising the water cut from f_{w1} to f_{w2}. This procedure is equivalent to combining layers between $P(k_1)$ and $P(k_2)$ into a single layer with average properties and allowing them to move with the same velocity, as given by Eq. 41.

Illustrative Example

The permeability data reported by Warren and Cosgrove[7] are used in this example. Twenty six points of permeability and corresponding thickness are rearranged in descending order and fitted to the log-normal distribution using the method of least squares (see Appendix B). The values of k_m and σ_k were found to be 257.465 md and 0.7937, respectively. The standard deviation σ_k corresponds to a value of V_{DP} of 0.5478. **Figs. 11 and 12** show the fitting of data on normal and probability plots, respectively. The value of M of 2.04 as used by Warren and Cosgrove is used to estimate the performance. Values of τ and R are calculated at the same values of F_{wo} reported by Warren and Cosgrove. The performance was then calculated using the developed correlation charts. Values of Ψ and ϕ are obtained from the charts for $V_{\mathrm{DP}}=0.55$ (between 0.5 and 0.6) for the corresponding values of F_{wo}/M. The dimensionless time τ and vertical coverage R are then calculated using Eqs. 20 and 21. The results of the calculations are shown in **Table 1** together with results from computer

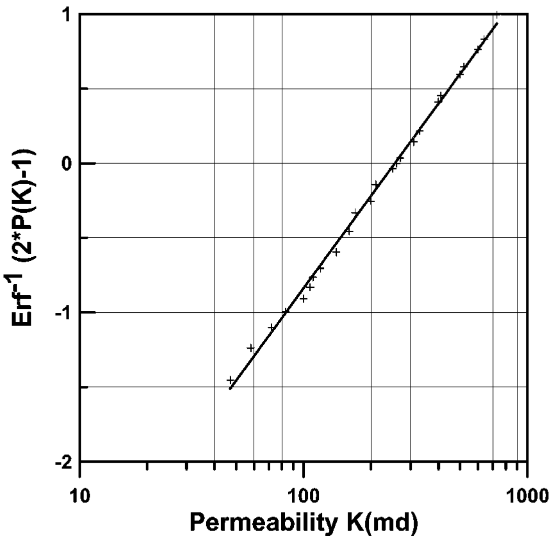

Fig. 12–Fitting of permeability distribution, probability graph.

calculations and those of Warren and Cosgrove. Table 1 shows the agreement between the results of this model and those obtained by Warren and Cosgrove. The small differences are due to the fact that Warren and Cosgrove used a graphical procedure to obtain values for $P(k)$ and $F(k)$ from probability plots of log(k). According to Eq. A-1, $F(k)$ is plotted parallel to $P(k)$ with $F(k)=0.5$ at the value $[k_m \exp(\sigma_k^2)]$. Values of k, $P(k)$ and $F(k)$ are substituted into Eqs. 11 to 13 to obtain the performance. It is to be noted that since the mobility ratio M is very close to 2 (2.04), the performance of the system can be obtained from Fig. 3 for a value of V_{DP} between 0.5 and 0.6.

Discussion

All the models for predicting the waterflooding performance in stratified reservoirs assume piston-like displacement with no saturation gradient behind the displacement front in the individual layers. Because of the variation of permeabilities and hence the velocities in the different layers, a step-wise variation in fractional flow and average saturation is established across the flow system between injection and production faces. The application of the material balance results in equations for the velocity of the displacement front in the different layers (corresponding to a given cross-sectional average saturation) similar to that for the BL displacement, i.e.,

$$dX_D/d\tau = df_w/dS_D. \tag{42}$$

TABLE 1–PERFORMANCE CALCULATIONS ($M=2.04, V_{\mathrm{DP}}=0.5478$)

F_{wo}	τ_w [a]	τ_m [b]	τ_c [c]	R_w	R_m	R_c
0.51	0.278	0.267	0.268	0.234	0.228	0.227
0.92	0.369	0.423	0.361	0.293	0.320	0.278
1.88	0.783	0.772	0.732	0.472	0.467	0.454
3.10	1.17	1.166	1.154	0.588	0.582	0.581
4.75	1.62	1.617	1.621	0.682	0.675	0.662
7.27	2.18	2.171	2.0	0.764	0.755	0.742
11.60	2.85	2.896	2.802	0.825	0.826	0.822
21.00	3.76	3.963	4.27	0.873	0.891	0.894
39.00	5.076	5.230	4.904	0.927	0.934	0.923
99.00	7.407	7.420	7.35	0.974	0.969	0.963
249.00	20.8	9.944	13.78	—	0.986	—

[a] w = Warren-Cosgrove model.
[b] m = current model–computer results.
[c] c = current model–correlation chart.

This equation can be integrated to give Eq. 37 only if f_w is a function of S_D only, i.e., time invariant. If f_w is time dependent as in the case of BL displacement with capillary effect taken into consideration, the velocity of the displacement front is not constant for each saturation and no analytical solution in closed form is possible.

The form of the fractional flow formula in stratified reservoirs depends on the assumption concerning communication between layers. In communicating models such as those of Hiatt[6] and Warren and Casgrove,[7] the value of f_w at a given horizontal location remains constant as this location is advancing in the system from the inlet to the outlet face. In other words, the fractional flow relation $f_w - S_D$ is independent of the location of the displacement front. On the other hand, in the DP model,[1] where noncommunication is assumed, the fractional flow formula depends on the location of the displacement front and it varies with time as the displacement front advances. The equations given by DP are only valid at the outlet face, i.e., as a given saturation zone reaches the producing well. Values of f_w and pseudorelative permeabilities calculated from these equations can not be used at other locations in the reservoir. This case is analogous to the case of BL displacement including the capillary pressure term in the $f_w - S_D$ curve. In such a case, the fractional flow is not a sole function of the water saturation S_D but depends on the saturation gradient $(\partial S/\partial X)$ which is a function of time. It is therefore clear that extending the frontal advance theory in the sense of BL displacement to stratified systems is only valid for communicating systems with complete crossflow between layers. Pande *et al.*[12] tried to extend this theory using the DP model (noncommunicating) but found agreement only for unit mobility ratio. At other mobility ratios, the velocities of the individual saturations were not constant with time as would be expected from Eq. 42 with a time-dependent f_w.

In BL displacement, the phenomenon of multiple-valued saturation profile is observed for almost all values of mobility ratio. This problem is handled by introducing a saturation shock or discontinuity that satisfies the material balance concept. The case of multiple-valued saturation profile is never met in the DP model. The displacement is always faster in the layers of higher permeabilities. In the case of communicating systems, the phenomenon of multiple-valued saturation profiles is only observed for mobility ratios less than unity, depending on the degree of heterogeneity of the reservoir (V_{DP} or σ_k). In such cases, the fractional flow curve possesses points of inflection and two different saturations move with the same velocity as can be seen from Fig. 9. This problem is handled by the same way as in the BL method. A number of layers (saturations) are lumped together and made to move with the same velocity. It is obvious from Fig. 9 that this phenomenon is less likely to occur in highly heterogeneous reservoirs. The fact that such a phenomenon is not observed with the DP model confirms the idea that the BL frontal advance theory is not applicable for this model.

As stated before, the factor R calculated by Eq. 15 represents vertical sweep efficiency (coverage) and its value reaches a maximum of unity when water breakthrough occurs at the layer with minimum permeability. If the distribution is untruncated log-normal, the minimum permeability would be zero which requires infinite time to reach water breakthrough. Since the displacement is assumed piston-like, the obtained coverage at any stage of displacement is usually multiplied by the ultimate recoverable oil saturation ΔS or $(1 - S_{wi} - S_{or})$. This ultimate recovery from swept zones can only be achieved after the injection of infinite pore volume of the displacement fluid and would be the same regardless of the mobility ratio of the displacement process. To be more accurate, the coverage R should rather be multiplied by the displacement efficiency which is determined by drawing a tangent to the $f_w - S_w$ curve with slope of $1/\tau$. The displacement efficiency at breakthrough depends on the mobility ratio and increases after breakthrough to reach the ultimate recovery factor of $(1 - S_{wi} - S_{or})$ at infinite time.

For favorable mobility ratios $(M<1)$, the displacement efficiency at breakthrough is high and reaches the ultimate recovery factor at a fast rate. So using the ultimate recovery factor of

$(1-S_{wi}-S_{or})$ for the displacement efficiency in this case does not introduce a serious error. Of course, the coverage should also be multiplied by the areal sweep efficiency of the flood pattern. On the other hand, for unfavorable mobility ratios $(M>1)$, the displacement efficiency at breakthrough is low and increases at a slower rate as displacement is continued to reach the ultimate recovery factor of $(1-S_{wi}-S_{or})$. The use of the ultimate recovery factor for displacement efficiency in this case would introduce a significant error. This explains the large difference in vertical sweep efficiency calculated by this method as compared to that obtained from reservoir simulation. This error will tend to be narrow as larger volumes of the displacement fluid are injected.

Conclusions

- A solution in closed form is developed for estimating the performance of waterflooding in communicating stratified reservoirs with log-normal permeability distribution. The equations, which include error functions and inverse error functions, are easily programmable for fast and accurate computations.
- The parameters in the developed equations can be combined in such a way to permit construction of a single correlation chart for each equation. These charts can be used for engineering calculations. Enlarged charts should be used for better accuracy.
- The developed linear t model assumes piston-like displacement. The vertical sweep efficiency should be multiplied by the displacement efficiency. For favorable mobility ratios $(M<1)$, the displacement efficiency may be replaced by the ultimate oil recovery.
- Pseudorelative permeability functions were also derived for the stratified system with the log-normal permeability distribution. These are only functions of saturation and heterogeneity. Such functions may be used in reservoir simulation to reduce the dimensionality of the simulator by averaging the properties in the vertical direction.
- At low mobility ratios (less than unity) the phenomenon of multiple-valued saturation profiles is observed. This problem is treated in a way similar to the BL saturation discontinuity.

Nomenclature

A = area L^2, ft^2 (m^2)
f_w = fractional flow of water, dimensionless
F_{wo} = water oil ratio, dimensionless
$F(k)$ = distribution function for first moment of permeability distribution, dimensionless
$h(k)$ = formation thickness with permeability greater than k, L, ft (m)
h_D = fraction of total thickness with permeability greater than K, dimensionless
h_t = total formation thickness, L, ft (m)
k = absolute horizontal permeability, L^2, md (μm^2)
k_m = mean of log-normal permeability distribution, L^2, md (μm^2)
k_{ro}^0 = oil relative permeability at irreducible water saturation, dimensionless
$\tilde{k}_{ro}$ = pseudorelative permeability for oil, dimensionless
k_{rw}^0 = water relative permeability at residual oil saturation, dimensionless
$\tilde{k}_{rw}$ = pseudorelative permeability for water, dimensionless
L = length, L, ft (m)
M = mobility ratio, dimensionless
$P(k)$ = distribution function of permeability, dimensionless
Q = flow rate, L^3/t, bbl/d (m^3/s)
R = vertical coverage, dimensionless
s = dimensionless formation capacity
S_D = dimensionless water saturation
S_{or} = residual oil saturation, fraction
S_w = water saturation, fraction
S_{wi} = initial water saturation, fraction
ΔS = displaceable oil saturation, fraction
t = time, t, d (s)
V_{DP} = Dykstra-Parsons variation coefficient, dimensionless
x = F_{wo}/M, dimensionless
X_D = dimensionless distance of the displacement front
μ = viscosity, m/Lt, cp (Pa·s)
σ_k = standard deviation of $P(k)$
τ = dimensionless time
Φ = function defined by Eq. 19, dimensionless
Ψ = function defined by Eq. 18, dimensionless

Subscripts

i = initial, irreducible
D = dimensionless
k = permeability
m = mean
o = oil
r = relative
t = total
w = water

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Appendix A: Derivation of the Model Equations

From the properties of the log-normal distribution, the function F defined by Eq. 8 has a log-normal distribution with mean $[K_m \exp(-\sigma_k^2)]$ and standard deviation σ_k. So,

$$F(k)=0.5+0.5\,\mathrm{erf}\left[\frac{\ln(k/k_m)-\sigma_k^2}{\sqrt{2}\sigma_k}\right], \tag{A-1}$$

from which

$$\ln(k/k_m)=\sigma_k^2+\sqrt{2}\sigma_k\,\mathrm{erf}^{-1}(2F-1), \tag{A-2}$$

and

$$(k/k_m)=\mathrm{Exp}\{\sigma_k^2+\sqrt{2}\sigma_k\,\mathrm{erf}^{-1}(2F-1)\}. \tag{A-3}$$

Since $f_w=F_{wo}/(1+F_{wo})$ and $s=1-F$, Eq. 11 can be written as

$$\frac{F_{wo}}{1+F_{wo}}=\frac{M(1-F)}{M(1-F)+F}, \quad \text{(A-4)}$$

from which

$$F=\frac{M}{M+F_{wo}}. \quad \text{(A-5)}$$

Differentiating f_w w.r.t. S_D

$$\frac{df_w}{dS_D}=\frac{df_w}{ds}\frac{ds}{dF}\frac{dF}{dP}\frac{dP}{dS_D}, \quad \text{(A-6)}$$

from Eqs. 6 and 8

$$dP(k)/dS_D=ds/dF=-1 \quad \text{(A-7)}$$

and from Eq. 11

$$\frac{df_w}{ds}=\frac{M}{[1+(M-1)s]^2}. \quad \text{(A-8)}$$

Also

$$\frac{dF}{dP}=\frac{\frac{dF}{d\ln(k/k_m)}}{\frac{dP}{d\ln(k/k_m)}}=\frac{k}{k_m}\exp(-0.5\sigma_k^2). \quad \text{(A-9)}$$

Substituting in Eq. A-6 we get

$$\frac{df_w}{dS_D}=\frac{M\exp[0.5\sigma_k^2+\sqrt{2}\sigma_k\,\mathrm{erf}^{-1}(2F-1)]}{[M+(1-M)F]^2}. \quad \text{(A-10)}$$

By substitution of Eqs. 12 and 13 we get

$$\tau=\frac{[M+(1-M)F]^2}{M}\mathrm{Exp}[-0.5\sigma_k^2-\sqrt{2}\sigma_k\,\mathrm{erf}^{-1}(2F-1)], \quad \text{(A-11)}$$

and

$$R=0.5-0.5\,\mathrm{erf}\left[\mathrm{erf}^{-1}(2F-1)+\frac{\sigma_k}{\sqrt{2}}\right]+\frac{\tau}{1+F_{wo}}. \quad \text{(A-12)}$$

Expressing the value of F in terms of water-oil ratio from Eq. A-5 and rearranging we get

$$\tau=\frac{[1+F_{wo}]^2}{M(1+x)^2}\mathrm{Exp}\left[-0.5\sigma_k^2-\sqrt{2}\sigma_k\,\mathrm{erf}^{-1}\left(\frac{1-x}{1+x}\right)\right], \quad \text{(A-13)}$$

and

$$R=0.5-0.5\,\mathrm{erf}\left[\mathrm{erf}^{-1}\left(\frac{1-x}{1+x}\right)+\frac{\sigma_k}{\sqrt{2}}\right]+\frac{\tau}{1+F_{wo}}, \quad \text{(A-14)}$$

where

$$x=\frac{F_{wo}}{M}. \quad \text{(A-15)}$$

Appendix B: Fitting Permeability Data to Log-Normal Distribution

The log-normal distribution of permeability is given by Eq. 1

$$p(k)=0.5+0.5\,\mathrm{erf}\left[\frac{\ln(k/k_m)}{\sqrt{2}\sigma_k}\right], \quad \text{(B-1)}$$

where $P(k)$ is taken as the relative cumulative thickness (fraction of the total thickness) with permeability less than K.

Eq. A-1 may be rearranged as

$$\mathrm{erf}\left[\frac{\ln(k/k_m)}{\sqrt{2}\sigma_k}\right]=2P(K)-1, \quad \text{(B-2)}$$

or

$$\left[\frac{\ln(k/k_m)}{\sqrt{2}\sigma_k}\right]\text{-}\mathrm{erf}^{-1}[2P(K)-1], \quad \text{(B-3)}$$

which may be written as

$$\ln k=\ln k_m+\sqrt{2}\sigma_k\,\mathrm{erf}^{-1}[2P(k)-1], \quad \text{(B-4)}$$

which is considered in the form

$$y=a+bx, \quad \text{(B-5)}$$

with $y=\ln k$, $a=\ln k_m$, $b=\sqrt{2}\sigma_k$, and $x=\mathrm{erf}^{-1}[2P(k)-1]$.

Values of x for the different data points are computed by evaluating $\mathrm{erf}^{-1}[2P(k)-1]$ and Eq. B-4 is fitted to a straight line by the least squares method. The slope of the straight line is $\sqrt{2}\sigma_k$ and the intercept is $(\ln k_m)$.

The error function erf(x) is defined as

$$\mathrm{erf}(x)=(2/\sqrt{\Pi})\int_0^x e^{-z^2}dz. \quad \text{(B-6)}$$

This function may be computed by numerical integration using Simpsons rules. It may also be evaluated using the algorithm given by Press *et al.*[14]

The inverse error function $\mathrm{erf}^{-1}(x)$ is evaluate using the Newton-Raphson method; let

$$\mathrm{erf}^{-1}(x)=y,$$

then

$$\mathrm{erf}(y)=x,$$

and

$$g(y)=\mathrm{erf}(y)-x=0. \quad \text{(B-7)}$$

A value of y is assumed and the function $g(y)$ is evaluated. If it is less than a specified tolerance ϵ, then y is accepted as the value for $\mathrm{erf}^{-1}(x)$. If not, a new value of y is obtained according to the Newton-Raphson scheme

$$y_{\text{new}}=y_{\text{old}}-[g(y)/g'(y)], \quad \text{(B-8)}$$

where

$$g'(y)=(2/\sqrt{\Pi})\exp(-y^2). \quad \text{(B-9)}$$

The process is continued until convergence is obtained. An initial guess for the value of y may be taken as $x/2$.

SI Metric Conversion Factors

bbl	×	1.589 873	E−01	=	m^3
cp	×	1.0*	E−03	=	Pa·s
ft	×	3.048*	E−01	=	m
ft^2	×	9.290 304*	E−02	=	m^2

*Conversion factors are exact.

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